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Inversion of Block Tridiagonal Matrices

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Abstract

We have implemented a prototype of the parallel block tridiagonal matrix inversion algorithm presented by Stig Skelboe [Ske09] using C# and the Microsoft.NET platform. The performance of our implementation was measured using the Mono runtime on a dual Intel Xeon E5310 (a total of eight cores) running Linux. Within the context of the software and hardware platform, we feel our results support Skelboes theories. We achieved speedups of 7.348 for inversion of block tridiagonal matrices, 7.734 for LU-factorization, 7.772 for minus – matrix – inverse-matrix multiply ($-\mathrm{A} \cdot \mathrm{B}^{-1}$), and 7.742 for inversion of matrices.

Resumé

Vi har implementeret en prototype i C# og Microsoft.NET af den parallelle algoritme til invertering af blok-tridiagonale matricer præsenteret af Stig Skelboe [Ske09]. Vi har foretaget ydelsesmålinger af vores prototype ved hjælp af Mono på en dual Intel Xeon E5310 (i alt otte kerner), der kører Linux. Vi mener at have understøttet Skelboes teorier, indenfor rammerne af platformen. Vores resultater inkluderer speedups på optil 7,348 for invertering af blok-tridiagonale matricer, 7,734 for LU-faktorisering, 7,772 for minus – matrix – invers-matrix multiplikation ($-\mathrm{A} \cdot \mathrm{B}^{-1}$), and 7,742 for invertering matricer.

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1 Introduction

Performing calculations in parallel is of increasing importance due to the stagnation in processor speed and increase in the number of processors and cores in computers. In the article [Ske09], Stig Skelboe presents a theoretical solution to the parallelization of inversion of a block tridiagonal matrix.

In this report we have documented our efforts to test Skelboes theories in practice through a prototype framework developed in C# on the Microsoft.NET platform. The experiments were performed on an eight core machine running Linux using the Mono runtime.

While we do reiterate key parts of Skelboes article related to our work, it may be beneficial for the reader to have copy at hand. A general knowledge of object oriented design and the challenges of parallel programming is also suggested.

All the source code, full set of measurement results, and the source material for this report are available online at http://github.com/egil/Inversion-of-Block-Tridiagonal-Matrices or on the accompanying compact disc.

2 Theory and description of the algorithms

In this section we will take a closer look at the theory and algorithms presented in [Ske09], and the modifications and derivations we have made. We do not recount [Ske09] in its entirety. The content of this section is the basis for the implementation presented in section 3 on page 18.

2.1 Theory

Given a quadratic block tridiagonal matrix, $\mathbf{A} \in (\mathbb{R}^{n\times m})^{N\times N}$, as in figure 1, where the diagonal blocks $\mathbf{a}_{ii}, i = 1 \ldots N$, of $\mathbf{A}$ themselves are quadratic, our problem is to calculate the part of the inverse, $\mathbf{G}=\mathbf{A}^{-1}$, of the same shape, size, and non-zero pattern as $\mathbf{A}$, using a parallel algorithm presented in [Ske09] which is based on the formulae deduced in [PSH+08].

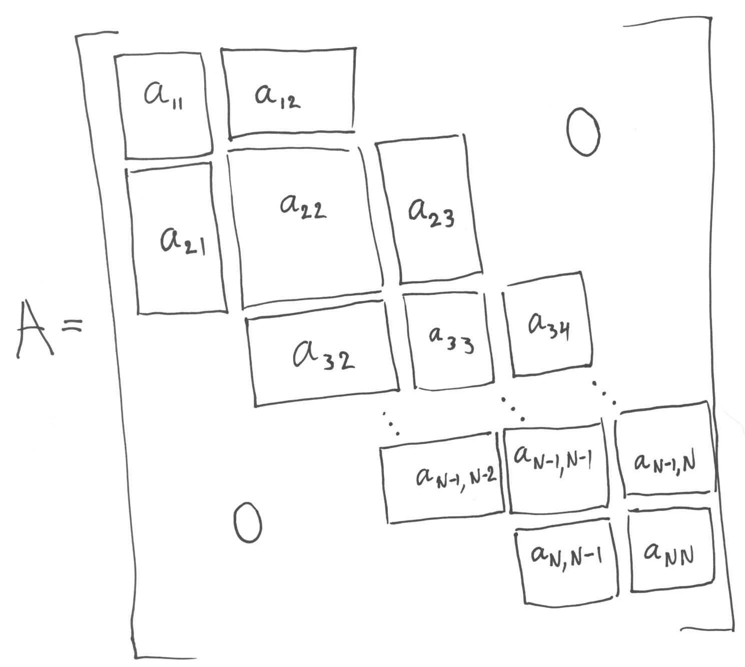


Figure 1: A sketch of how the quadratic block tridiagonal matrix, A, could look. The shape of the blocks is indicated.

The algorithm consists of two independent sweeps, (1) and (3), each producing intermediate results used in the final calculations, (5) and (6), all four operating on the blocks of $\mathbf{A}$. The two sweeps are, as presented in [Ske09], here with a slight change in notation:

A downward sweep,

$$\boldsymbol{c}_i^L = -\boldsymbol{a}_{i+1,i}(\boldsymbol{d}_{ii}^L)^{-1}, i = 1, 2, 3, \ldots, N-1 \tag{1}$$

$$\text{where } \boldsymbol{d}_{11}^L = \boldsymbol{a}_{11}$$

$$\boldsymbol{d}_{ii}^L = \boldsymbol{a}_{ii} + \boldsymbol{c}_{i-1}^L \boldsymbol{a}_{i-1,i} \tag{2}$$

An upward sweep,

$$\boldsymbol{c}_i^R = -\boldsymbol{a}_{i-1,i}(\boldsymbol{d}_{ii}^R)^{-1}, i = N, N-1, N-2, \dots, 2 \tag{3}$$

$$\text{where } \boldsymbol{d}_{NN}^R = \boldsymbol{a}_{NN}$$

$$\boldsymbol{d}_{ii}^R = \boldsymbol{a}_{ii} + \boldsymbol{c}_{i+1}^R \boldsymbol{a}_{i+1,i} \tag{4}$$

The final calculations for the blocks $\boldsymbol{g}_{ij}$ of $\boldsymbol{G} = \boldsymbol{A}^{-1}$ are,

$$\boldsymbol{g}_{ii} = (-\boldsymbol{a}_{ii} + \boldsymbol{d}_{ii}^L + \boldsymbol{d}_{ii}^R)^{-1}, i = 1, 2, 3, \dots, N \tag{5}$$

$$\begin{aligned} \boldsymbol{g}_{ij} &= \boldsymbol{g}_{ii} \cdot \boldsymbol{c}_{i+1}^R \cdot \boldsymbol{c}_{i+2}^R \cdots \boldsymbol{c}_j^R \text{ for } i < j \\ \boldsymbol{g}_{ij} &= \boldsymbol{g}_{ii} \cdot \boldsymbol{c}_{i-1}^L \cdot \boldsymbol{c}_{i-2}^L \cdots \boldsymbol{c}_j^L \text{ for } i > j \end{aligned} \tag{6}$$

2.2 Tiling

The formulae presented in section 2.1 on the previous page work on a per block basis. In order to distribute the work over several processors, each block, $\mathbf{a}_{ij}$, of $\mathbf{A}$, is divided into a number of tiles $\mathfrak{a}_{kl}$, making $\mathbf{a}_{ij}$ a block matrix on its own.

Figure 2 on the following page illustrates how a block $\mathbf{a}_{ij}$ is tiled into a block matrix. Note that the rightmost and bottom tiles may not be quadratic due to the tile size n not dividing the size of the block O and P.

It is important to use the same tile size, n, and tiling strategy on all blocks in order for the tiled matrix operations to be meaningful.

Failing to do this, the formulae will be rendered incorrect at best, because numbers in the wrong positions in the original matrix will be operated on. Most likely though, the tiled matrix operations will fail on account of tiles not matching size requirements when for example multiplying or adding two tiles together.

2.3 The matrix operations

In this section we will look at some of the algorithms used in our calculations of the formulae presented in section 2.1 on the previous page.

We have modified the tiled matrix – inverse-matrix multiply algorithm from [Ske09] section 4 to incorporate the minus operation, which is always used in calculation of formulae (1) and (3), resulting in a tiled minus – matrix – inverse-matrix multiply algorithm.

When calculating formula (1), we first LU-factorize the term $\mathbf{d}_{ii}^L$ before applying the tiled minus – matrix – inverse-matrix multiply algorithm.

Also, when calculating formula (2), we use a combined tiled addition and tiled matrix multiplication named tiled plus – multiply when computing $\mathbf{d}_{\mathrm{ii}}^L$.

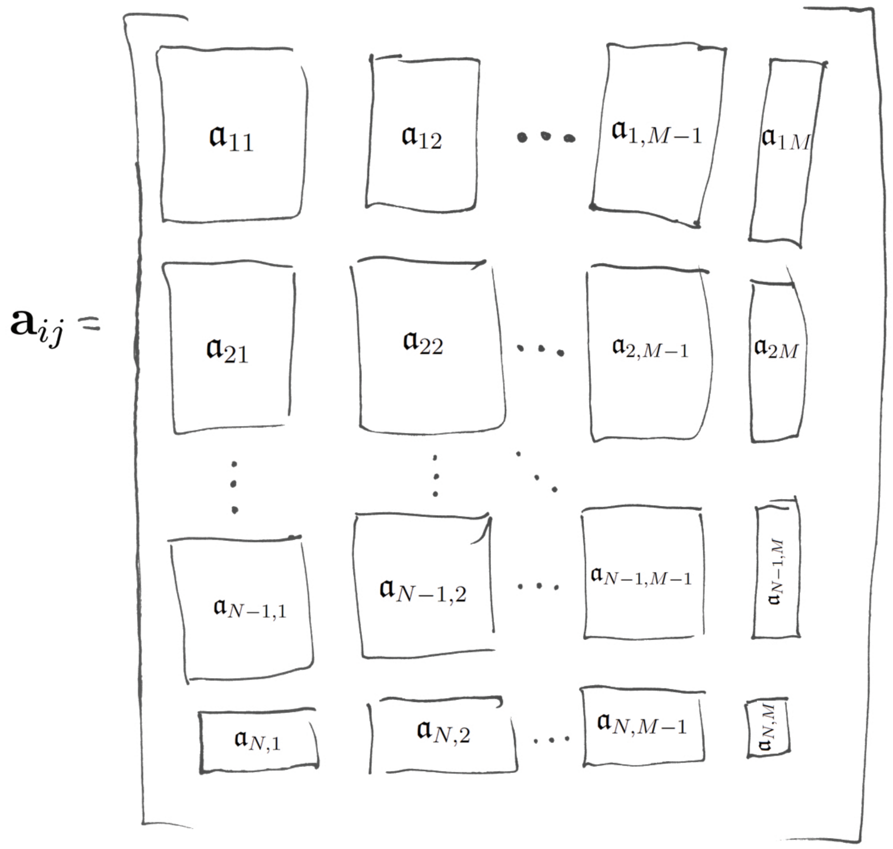


Figure 2: The tiling of a block $\mathbf{a}_{ij}$ of size $O{\times}P$ into a block matrix of size $N{\times}M$ where $N{=}\lceil O/n\rceil$, $M{=}\lceil P/n\rceil$, and n is the chosen tile size. The tiles $\mathfrak{a}_{kl}, k{<}N, j{<}M$ are of size $n{\times}n$. Unless n divides both N and M, the rightmost tiles, $\mathfrak{a}_{kM}$ are of size $n{\times}(P{-}\left\lfloor \frac{P}{n} \right\rfloor)$, and the bottom tiles $\mathfrak{a}_{Nl}$ are of size $(O{-}\left\lfloor \frac{O}{n} \right\rfloor){\times}n$, making $\mathfrak{a}_{NM}$ of size $(O{-}\left\lfloor \frac{O}{n} \right\rfloor) \times (P{-}\left\lfloor \frac{P}{n} \right\rfloor)$.

The same tiled matrix operations are used in (3) and (4).

In the calculations of the diagonal elements, $\mathbf{g}_{ii}$, in (5), $-\mathbf{a}_{ii}{+}\mathbf{d}^{L}_{ii}{+}\mathbf{d}^{R}_{ii}$ is combined into one tiled minus – plus – plus operation, LU-factorized and then inverted.

The last formula (6) is a series of tiled matrix-matrix multiplications.

Plus – multiply, minus – plus – plus, and matrix multiplication are all delightfully parallel; there are no dependencies among the tiles, and we will not go into much detail with these.

2.3.1 LU-factorization

The psuedocode in figure 3 on the following page is from [Ske09]. Figure 4 on page 13 illustrates how tiled LU-factorization is carried out.

To parallelize the algorithm in figure 3, the computational kernels are identified as follows:

```
1  $L_{11}U_{11} = A_{11}^{(0)}$
2  for $k = 1 : (N-1)$
3      for $i = (k+1) : N$
4          $L_{ik} = A_{ik}^{(k-1)} U_{kk}^{-1}$
5          $U_{ki} = L_{kk}^{-1} A_{ki}^{(k-1)}$
6      end
7      for $i = (k+1) : N$
8          for $j = (k+1) : N$
9              $A_{ij}^{(k)} = A_{ij}^{(k-1)} - L_{ik}U_{kj}$
10         end
11     end
12     $L_{k+1,k+1}U_{k+1,k+1} = A_{k+1,k+1}^{(k)}$
13 end
```

Figure 3: Tiled LU-factorization

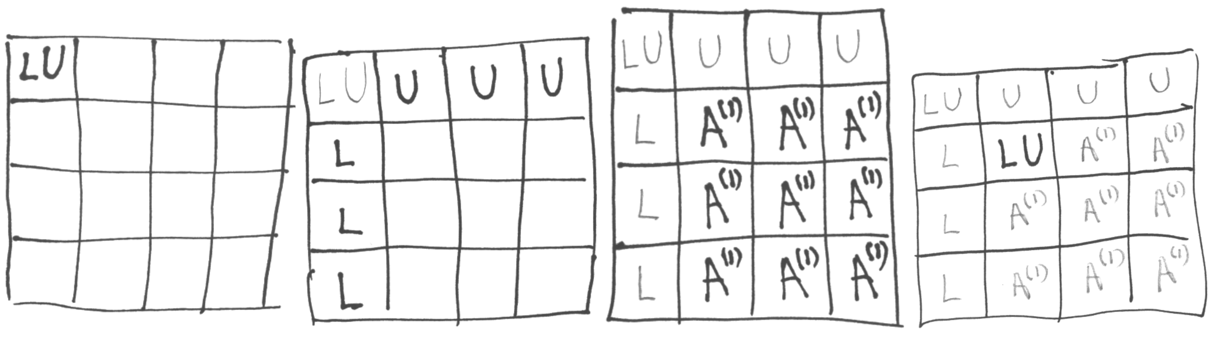
- Line 1 and 12 is an LU-factorization
- In line 4 there is an matrix inverse-matrix multiply
- In line 5 there is an inverse-triangular-matrix – matrix multiply
- Line 9 has a matrix – matrix multiply add

The dependencies, which are visible from the pseudocode, are summarized in the following `DependsOn` function, reproduced from [Ske09] section 3.2.2. An operation is never started before its dependencies are satisfied.

`DependsOn`$(L_{ii}U_{ii}) \rightarrow \{A_{ii}^{(i-1)}\}$
`DependsOn`$(A_{ij}^{(k)}) \rightarrow \{A_{ij}^{(k-1)}, L_{ik}, U_{kj}\}$
`DependsOn`$(L_{ij}) \rightarrow \{A_{ij}^{(j-1)}, U_{jj}\}$
`DependsOn`$(U_{ij}) \rightarrow \{L_{ii}, A_{ij}^{(i-1)}\}$

A scheduling of the operations is presented in [Ske09] section 3.2.2, basically interleaving the for loops of the algorithm to continually free many dependencies[1]. The naive way to schedule the computational kernels results in periods where only one operation is runnable. To eliminate this when utilizing more cores, we will use the modified pipelined diagonal sweep elimination algorithm exemplified in [Ske09], figure 4.

[1]Finding the optimal scheduling is an NP-complete problem, so we will limit ourselves to the schedule suggested in [Ske09].



(a) (b) (c) (d)

Figure 4: This illustrates how the psuedocode in figure 3 on the previous page runs. a) Line 1: The upper left tile is LU-factorized using a standard matrix LU-factorization algorithm. b) Line 3-6: The first column and row of the L and U matrices respectively is calculated. c) Line 7-11: The sub matrix is updated. d) Line 12: The next diagonal element is LU-factorized (as in line 1) and the algorithm continues on the sub matrix resulting from removing the first column and row.

2.3.2 Minus – matrix – inverse-matrix multiply

We use the tiled matrix – inverse-matrix multiply algorithm from [Ske09] figure 6.

In order to avoid the iteration performing the unary minus operation on the resulting block matrix, we modified the presented algorithm by changing the sign of right hand side of $C_{i,j}$ (line 3 and 8) and changing the minus operation to a plus operation in line 5 in the calculation of $B_{i,j}^{(k+1)}$. The modified pseudocode for tiled minus — matrix — inverse-matrix multiply is shown in figure 5 on the following page. This combined algorithm assumes its second argument is an LU-factorized matrix.

The algorithm operates on each row independently; it calculates the first tile (line 3 and 8) and updates the rest of the row (line 5), and then recur ignoring the first tile. The second step operates in the same manner, moving from right to left.

To parallelize tiled minus – matrix – inverse-matrix multiply the computational kernels are identified from figure 5 on the next page:

- The calculation of a resulting tile – line 3 and 8 (in both step one and step two)
- The intermediate calculations – line 5 (in both step one and step two)

We have derived the dependencies between the computational kernels to be as follows (using the same notation style as in [Ske09]):

$\texttt{DependsOn}(B_{ik}) \rightarrow \{A_{ik}^{(k-1)}, U_{kk}\}$
$\texttt{DependsOn}(A_{ij}^{(k)}) \rightarrow \{A_{ij}^{(k-1)}, B_{ik}, U_{kj}\}$
$\texttt{DependsOn}(C_{i,j}) \rightarrow \{B_{i,j}^{(k)}, L_{j,j}\}$

```
1  for $i = 1 : N$
2    for $k = 1 : (N-1)$
3      $B_{ik} = A_{ik}^{(k-1)} U_{kk}^{-1}$
4      for $j = (k+1) : N$
5        $A_{ij}^{(k)} = A_{ij}^{(k-1)} - B_{ik} U_{kj}$
6      end
7    end
8    $B_{iN} = A_{iN}^{(N-1)} U_{NN}^{-1}$
9  end
```

$\mathbf{b} = \mathbf{a}\mathbf{u}^{-1}$

```
1  for $i = 1 : N$
2    for $k = 0 : (N-2)$
3      $C_{i,N-k} = -B_{i,N-k}^{(k)} L_{N-k,N-k}^{-1}$
4      for $j = (N-1-k) : -1 : 1$
5        $B_{ij}^{(k+1)} = B_{ij}^{(k)} + C_{i,N-k} L_{N-k,j}$
6      end
7    end
8    $C_{i1} = -B_{i1}^{(N-1)} L_{11}^{-1}$
9  end
```

$\mathbf{c} = \mathbf{b}\mathbf{l}^{-1}$

Figure 5: The modified psuedocode for tiled minus – matrix – inverse-matrix multiply. The original can be found in [Ske09], figure 6. Changes are: The sign of the right hand side of $C_{i,j}$ (line 3 and 8), the minus operator is changed to a plus operator in line 5 in the calculation of $B_{i,j}^{(k+1)}$.

$\texttt{DependsOn}(B_{ij}^{(k+1)}) \to \{B_{ij}^{(k)}, C_{i,N-k}, L_{N-k,j}\}$

Where A is the first argument to be multiplied with $D = LU$ before being subject to unary minus; B is the block resulting from the first step (figure 5, left side); $B_{ij}^{(k)}$ are the intermediate results from step two (figure 5, right side), with $B_{ij}^{(0)} = B_{ij}$; and $C_{i,N-k}$ are the tiles of the final result. $A_{ij}^{(k)}$, B_{ik}, $B_{ij}^{(k+1)}$, and $C_{i,N-k}$ may safely refer to the same block as they do not coincide in time and space.

We note that when the i'th row of step one has completed, the i'th row of step two can be calculated. This property makes pipelining the calculation of the rows between different steps straightforward.

As with LU-factorization a scheduling of the operations is suggested in [Ske09] figure 7 for the first step of tiled minus – matrix – inverse-matrix multiply. The scheduling of the second step is very similar, except that the rows are treated from right to left. We illustrate this scheduling in figure 6 on the following page.

2.3.3 Matrix Inverse

We use the tiled matrix inversion algorithm from [Ske09], figure 9, reproduced in figure 7 on the next page.

The algorithm works in much the same way as the tiled matrix – inverse-matrix multiply algorithm, except that it processes columns instead of rows.

In the first step, the algorithm runs from the top to the bottom of the matrix, exploiting the lower triangular nature of the intermediate result matrix to skip zero filled tiles above diagonal. This is illustrated in figure 8 on page 16.

1:	C_{i5}				
2:	$B_{i4}^{(1)}$				
3:	$B_{i3}^{(1)}$	C_{i4}			
4:	$B_{i2}^{(1)}$	$B_{i3}^{(2)}$			
5:	$B_{i1}^{(1)}$	$B_{i2}^{(2)}$	C_{i3}		
6:		$B_{i1}^{(2)}$	$B_{i2}^{(3)}$		
7:			$B_{i1}^{(3)}$	C_{i2}	
8:				$B_{i1}^{(4)}$	
9:					C_{i1}

Figure 6: The scheduling of row i of the second step of the tiled minus - matrix - inverse-matrix multiply algorithm, using the same notation as [Ske09] figure 7. In this example the block matrix has 5 columns. The figure is read line by line.

```
1  for $j = 1 : N$
2    for $k = 0 : (N - j - 1)$
3      $F_{k+j,j} = L^{-1}_{k+j,k+j} R^{(k)}_{k+j,j}$
4      for $i = (j + k + 1) : N$
5        $R^{(k+1)}_{ij} = R^{(k)}_{ij} - L_{i,k+j} F_{k+j,j}$
6      end
7    end
8    $F_{Nj} = L^{-1}_{NN} R^{(N-j)}_{Nj}$
9  end
```

$$\mathbf{f} = \mathbf{l}^{-1}\mathbf{r}$$

```
for $j = 1 : N$
  for $k = 0 : (N - 2)$
    $G_{N-k,j} = U^{-1}_{N-k,N-k} F^{(k)}_{N-k,j}$
    for $i = (N - 1 - k) : -1 : 1$
      $F^{(k+1)}_{ij} = F^{(k)}_{ij} - U_{i,N-k} G_{N-k,j}$
    end
  end
  $G_{1j} = U^{-1}_{11} F^{(N-1)}_{1j}$
end
```

$$\mathbf{g} = \mathbf{u}^{-1}\mathbf{f}$$

Figure 7: Tiled matrix inversion from [Ske09], figure 9. Note that i and j are interchanged to improve readability.

In the second step, the algorithm runs from the bottom all the way to the top of the matrix.

To parallelize the tiled matrix inverse algorithm, the computational kernels are identified as seen in figure 7:

- The calculation of a resulting tile – line 3 and 8 (in both step one and step two)
- The intermediate calculations – line 5 (in both step one and step two)

We have derived the dependencies between the computational kernels to be as follows:

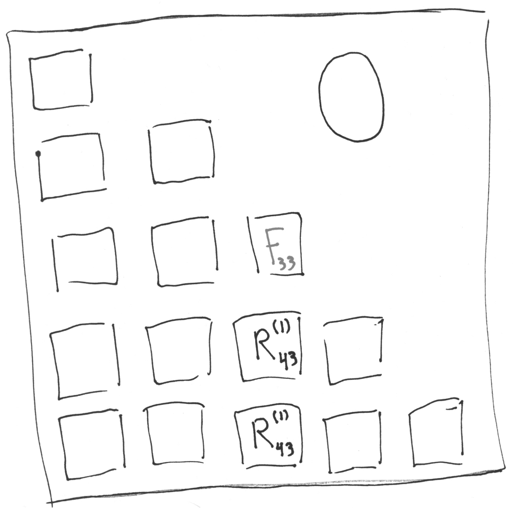


Figure 8: The second iteration of calculating column 3 of the first step of Inverse. The tiles above the diagonal are skipped. The example is of size 5×5

$\texttt{DependsOn}(F_{k+j,j}) \rightarrow \{L^{-1}_{k+j,k+j}, R^{(k)}_{k+j,j}\}$
$\texttt{DependsOn}(R^{(k+1)}_{ij}) \rightarrow \{R^{(k)}_{ij}, L_{i,k+j}, F_{k+j,j}\}$
$\texttt{DependsOn}(G_{i,j}) \rightarrow \{U^{-1}_{i,i}, F^{(k)}_{i,j}\}$
$\texttt{DependsOn}(F^{(k+1)}_{ij}) \rightarrow \{F^{(k)}_{ij}, U_{i,N-k}, G_{N-k,j}\}$

$F_{k+j,j}$, $R^{(k+1)}_{ij}$, $G_{N-k,j}$, and $F^{(k+1)}_{ij}$ do not refer to the same tiles at the same time, and can hence safely refer to same block matrix.

Similar to the pipelined properties of the parallel tiled matrix – inverse-matrix multiply algorithm, when the i'th column of step one is complete, the i'th column of step two can proceed.

As suggested in [Ske09] section 5, we have derived a schedule similar to that of the tiled matrix – inverse-matrix multiply algorithm. See figure 9 and 10 on the following page for an illustration of the schedule of step one and two respectively.

2.4 Utilizing the combined operations in block tridiagonal matrix inversion

When putting it all together, formula (1) becomes three operations:

- A plus multiply operation is used to calculate $\mathbf{d}^L_{ii}$ from (2)
- An LU-factorization of $\mathbf{d}^L_{ii}$ from (1)
- A minus – matrix – inverse-matrix multiply operation calculating $\mathbf{c}^L_{i-1}$ from (1)

$$\begin{array}{llllll}
1: & F_{11} & & & & \\
2: & R^{(1)}_{21} & & & & \\
3: & R^{(1)}_{31} & F_{21} & & & \\
4: & R^{(1)}_{41} & R^{(2)}_{31} & & & \\
5: & R^{(1)}_{51} & R^{(2)}_{41} & F_{31} & & \\
6: & & R^{(2)}_{51} & R^{(3)}_{41} & & \\
7: & & & R^{(3)}_{51} & F_{41} & \\
8: & & & & R^{(4)}_{51} & \\
9: & & & & & F_{51}
\end{array}$$

Figure 9: The scheduling of column 1 of the first step of the tiled matrix inverse using the same notation style as [Ske09], figure 7. This example has five rows. The scheduling of column j is obtained by removing the operations of the previous columns and subtracting $j-1$ from the time step, noted in all superscripts – and of course replacing 1 with j.

$$\begin{array}{llllll}
1: & G_{5j} & & & & \\
2: & F^{(1)}_{4j} & & & & \\
3: & F^{(1)}_{3j} & G_{4j} & & & \\
4: & F^{(1)}_{2j} & F^{(2)}_{3j} & & & \\
5: & F^{(1)}_{1j} & F^{(2)}_{2j} & G_{3j} & & \\
6: & & F^{(2)}_{1j} & F^{(3)}_{2j} & & \\
7: & & & F^{(3)}_{1j} & G_{2j} & \\
8: & & & & F^{(4)}_{1j} & \\
9: & & & & & G_{1j}
\end{array}$$

Figure 10: The scheduling of column j of the second step of the tiled matrix inverse using the same notation style as [Ske09], figure 7. This example has five rows. This step resembles those of minus – matrix – inverse-matrix multiply, the only difference is that it iterates over columns instead of rows.

Likewise, formula (3) is transformed into the same three operations (with R substituted for L).

Formula (5) also turns into three operations:

- A minus plus plus operation calculating $-\mathbf{a}_{ii}+\mathbf{d}^L_{ii}+\mathbf{d}^R_{ii}$.
- An LU-factorization of the result of the minus plus plus operation above.
- A calculation of the inverse using the LU-factorized result of $-\mathbf{a}_{ii}+\mathbf{d}^L_{ii}+\mathbf{d}^R_{ii}$.

Formula 6 on page 10 is calculated using matrix multiplication.

Since the final calculations depend on the upward and downward sweeps, these are calculated first.

3 Implementation

We have implemented the following components:

- A matrix and block tridiagonal matrix.
- Support functions to tile and untile matrices and block tridiagonal matrices.
- Mathematical functions on both matrices and tiled matrices. These include the following basic matrix operations: addition, subtraction, multiplication, and unary minus. We also have matrix inverse and LU factorization and for the purpose of our project we added some combined operations to speed things up: “plus multiply” ($\mathbf{a} + \mathbf{b} * \mathbf{c}$), “minus plus plus” ($-\mathbf{a} + \mathbf{b} + \mathbf{c}$), “minus matrix inverse matrix multiply” ($-\mathbf{a} * \mathbf{b}^{-1}$).
- Functions to compute the inverse of a block tridiagonal matrix as well as a tiled block tridiagonal matrix.
- A parallel version of all mathematical functions mentioned above.
- A parallel version of the function to compute the inverse of a tiled block tridiagonal matrix.

The matrix operations mentioned above only support calculations on `double`s. It would, however, be trivial to add support for complex numbers due to our use of C# support for generic programming.

All our code is supported by a unit test suite that covers the use cases we need. We do not guarantee a correct result in other cases, for example we do not do input validation which could prevent division by zero. This could happen if the input matrix is singular.

We also have a few support runtimes that help with testing and generating random datasets, which we will not go into in this paper.

3.1 Workflow

In the following section, we will look at how our parallel library works, how computational kernels are produced and assigned processor time, and how communication between the different parts of the system works.

The main players in our design are a **producer** of computational kernels, a **manager** that consumes the computational kernels and a number of **workers** (threads) that processes the computational kernels for the manager.

The producer A producer can be anything from a simple parallel matrix multiplication to something that represents a full collection of formulae – like the complete inversion of a tiled block tridiagonal matrix – as long as it produces only runnable computational kernels.

A producer is able to indicate to the manager that all computational kernels have been produced, or that it does not have any runnable computational kernels at the time of inquiry.

The manager The manager is responsible for starting a suitable number of workers, each running in their own thread, delivering computational kernels to them from its associated producer, and, when the producer is finished, terminate the workers and exit.

The worker The workers execute computational kernels and, once done, ask for more work. If there is no work available at a given time, the worker will take a break and wake up again when more work may be available again.

We have illustrated how these parts interact in figure 11.

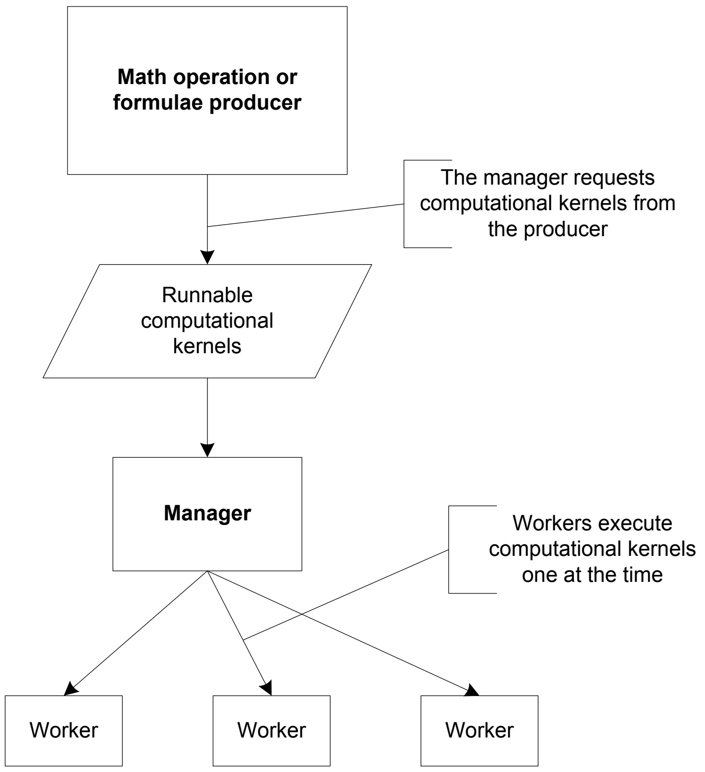
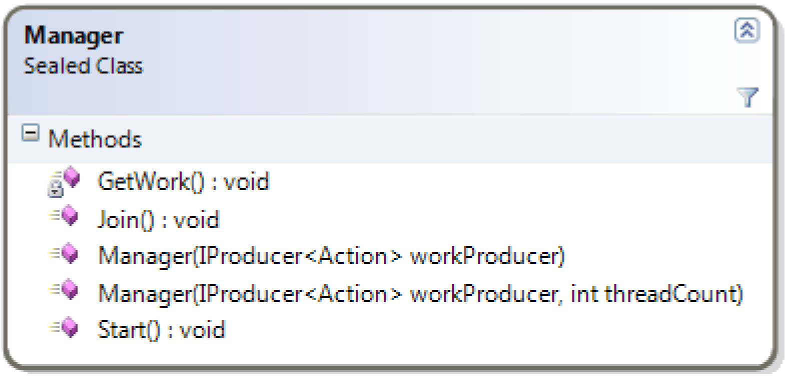

Math operation or formulae producer
The manager requests computational kernels from the producer
Runnable computational kernels
Manager
Workers execute computational kernels one at the time
Worker
Worker
Worker

Figure 11: Conceptual workflow of the parallel computations.



```
GetWork() : void
Join() : void
Manager(IProducer<Action> workProducer)
Manager(IProducer<Action> workProducer, int threadCount)
Start() : void
```

Figure 12: A class diagram of the `Manager` class.

3.2 Overall design

At the core, we tried to keep the code simple and easy to understand. To achieve this, we have utilized many of the build-in constructs in the .net framework. Examples of this include generics, which allowed us to have a single matrix class that is used both as a tiled matrix, block matrix and a standard matrix, and the build in support for the iterator pattern, which made the implementations of the different producers much cleaner.

For speed we tried our best to keep the code lock free, only employing one very fast spin wait lock in the code section where the manager requests computational kernels from a producer for its workers.

The computational kernels are represented by the .net `Action` class. An action encapsulates a method that takes no parameters and does not return a value.

Since a matrix operation cannot start before its data is ready, we created a class called `OperationResults` which encapsulates the data matrix and a bit matrix of the same size. Each bit in the bit matrix indicates whether the corresponding position in the data matrix is ready; the advantage of using a bit matrix compared to a table of boolean values is the smaller memory footprint.

This makes it possible for the result of one matrix operation to be the input of another matrix operation. Besides allowing us to chain matrix operations together, it also allows us to pipeline operations by starting a matrix operation before the preceding operation has fully completed.

3.3 Components

3.3.1 Manager (and workers)

As the class diagram in figure 12 shows, the `Manager` class can be instantiated with the number of desired workers (`threadCount`). If `threadCount` is omitted the `Manager` class defaults to the number of logical processors in the computer.

The `Start` method starts the worker threads and the `Join` method allows the caller to join on all the worker threads, thus blocking until all work is done.

The `GetWork` method is a private method where the workers exist.

The workers are represented by the build-in class `Thread`. The `Thread` class takes a method as its argument, and once started, the thread will keep running until it reaches the end of that method or is terminated from the outside. The method that is handed to our workers is `GetWork`.

A closer look at the GetWork method In figure 13 we illustrate the workflow of the `GetWork` method.

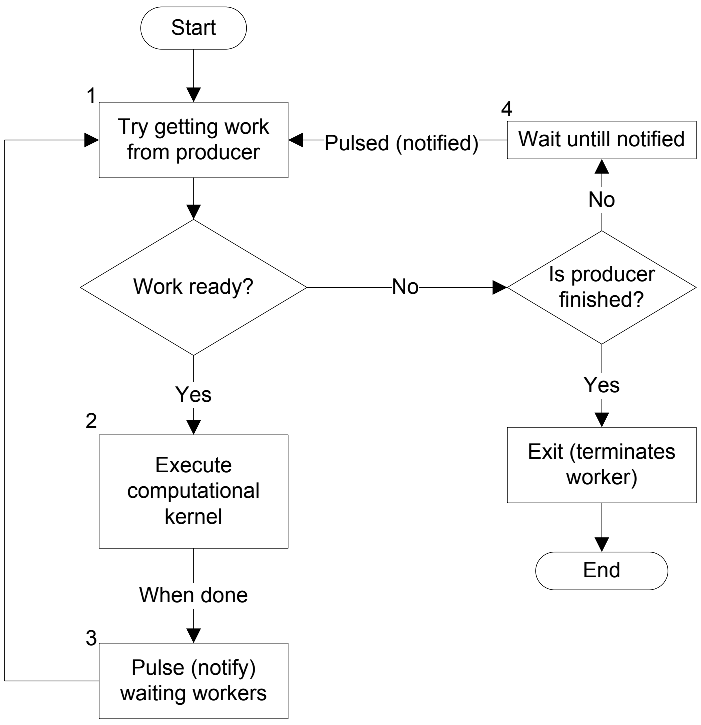


Figure 13: Workflow of the `GetWork` method.

We have to use a lock during `GetWork` when requesting a computational kernel (step 1 in figure 13), to ensure that a computational kernel is handed out only once. The time it takes the producer to either generate a ready computation kernel or signal the thread to wait is very short, so using a `SpinWaitLock` instead of a regular lock prevents a thread from blocking when the lock is contented. For a thread to enter a blocking state in .net, it would require a context switch to kernel mode, which is a very expensive operation. With a `SpinWaitLock` we stay in user mode with very little overhead as a result.

We do have a very subtle race condition in the `GetWork` workflow.

Consider this: Looking at figure 13, imagine Worker 1 asks the producer for work in step 1, but the producer does not return anything. Worker 1 then proceeds to check if the producer is finished (i.e. all of its computational kernels have been processed). If the producer is not finished Worker 1 has to enter a waiting state, where it is notified the next

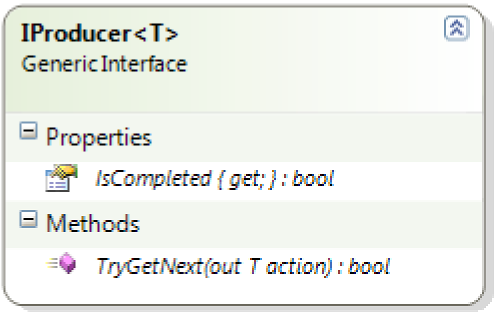


Figure 14: To represent a producer we have a generic interface called `IProducer<T>`, which every producer must implement. All the producers return Actions representing computational kernels.

time another worker finish processing a computational kernel (the logic being that there is never new runnable computational kernels available before a worker finishes one).

The trouble starts if Worker 2 finishes a computational kernel and then proceeds to step 3 where it notifies waiting co-workers. If it manages to signal its co-workers before Worker 1 is able to enter a waiting state, Worker 1 will miss the notification from Worker 2 and will have to wait until another co-worker finishes processing a computational kernel and notifies it.

The race condition will never result in a deadlock since the waiting is eventually interrupted by a timeout (set to five seconds). The worst case scenario is that the worker has to wait until another worker finishes a computational kernel and notifies it. We do not have any precise way of measuring how often this occurs. We are confident that we, with more time, could have solved this race condition.

3.3.2 `IProducer<T>`

The `IProducer<T>` generic class describes the interface that producers must adhere to if they want the `Manager` class to execute its computational kernels in parallel. Figure 14 shows the class diagram for the `IProducer<T>` interface.

The `IsCompleted` property is used to indicate whether or not the producer is exhausted. The `TryGetNext` method is used to request a new computational kernel from the producer. If the producer does not have a runnable computational kernel ready, `TryGetNext` will return false. If there is a runnable computational kernel available, `TryGetNext` will assign that computational kernel to the out action parameter and return true.

It is entirely up to the producer to decide when a computational kernel is runnable, which includes keeping track of dependencies between its computational kernels and data they are working on.

We decided to make the `IProducer<T>` class a generic class to allow us to have producers that would produce other producers. This is used in the `BlockTridiagonalMatrixInverse` class to produce individual matrix operations that in turn produce actions.

3.3.3 The matrix operation producer pattern

Generally, all the classes that implement the `IProducer<Action>` interface have the following components:

AbstractOperation an internal class used to represent an abstract operation. It encapsulates the indices of tiles in the matrix to perform a given operation on. It may also hold an operation type that indicates the mathematical operation to perform. An example of where an operation type is necessary is in the LU Factorization operation, where we need to keep track of whether we are calculating an A, L, U or an LU.

AbstractOperationGenerator a private method that returns an iterator over a sequence of abstract representations of computational kernels. This sequence is generated lazily through .nets iterator pattern, instead of having it hardcoded at compile time. This saves a lot of memory since only a fraction of the total abstract operations is in memory at any time.

OperationEnumerator the iterator returned by `AbstractOperationGenerator` is wrapped inside this class. This class has an internal buffer of abstract operations and allows a caller to retrieve an abstract operation matching a specified predicate. Internally, the producer will call a method called `Find` on this class to determine if there are any runnable abstract operations available. The workflow diagram in figure 15 on the following page illustrates this.

IsRunnable a method used to decide if a given abstract operation is runnable, that is, the required data is ready. This method is used as the predicate when querying the operation enumerator. `IsRunnable` is roughly equivalent to the function `DependsOn`. The more complex matrix operations use internal status tables to keep track of the progress. They are updated by the end of a computational kernel and read from the `IsRunnable` method.

GenerateAction a method that converts an abstract operation into an actual computational kernel.

This basic structure is exemplified in figure 16 on page 26.

3.3.4 The straightforward producers

In the category "straightforward producers" we have `Multiply`, `MinusPlusPlus`, `PlusMultiply`, `TileOperation`, `UntileOperation`, and `SimpleProducer`. `SimpleProducer` is a special producer that only produces one computational kernel, a computational kernel it receives as its argument. It is used in the `BlockTridiagonalMatrixInverse` producer to do a few simple copy operations of data where needed.

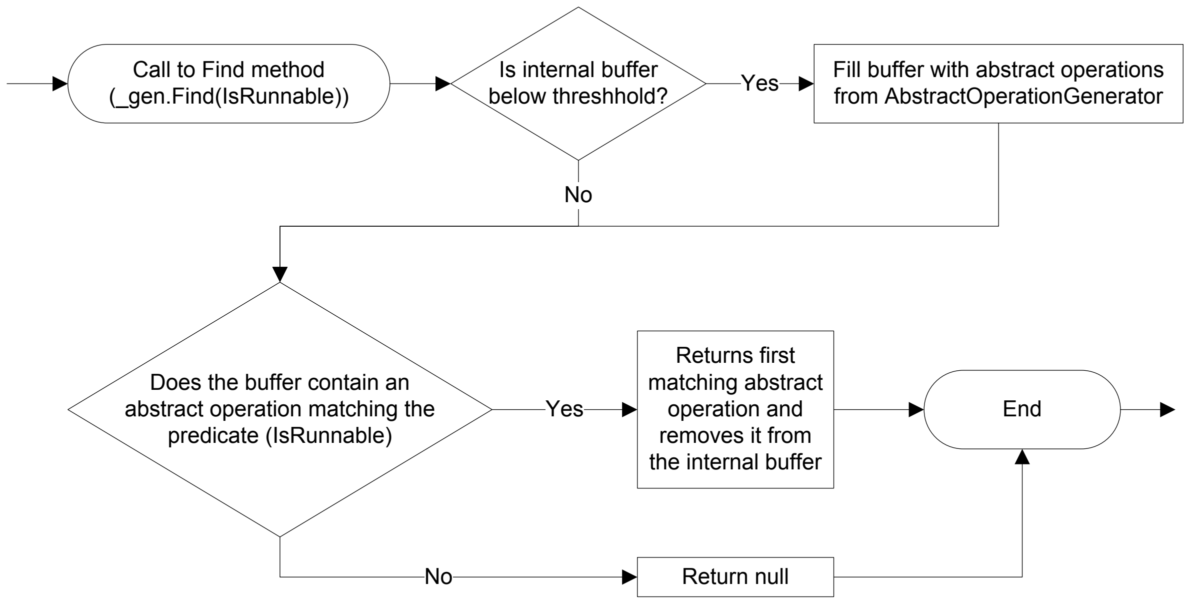

Call to Find method (_gen.Find(IsRunnable))
Is internal buffer below threshhold?
Yes
Fill buffer with abstract operations from AbstractOperationGenerator
No
Does the buffer contain an abstract operation matching the predicate (IsRunnable)
Yes
Returns first matching abstract operation and removes it from the internal buffer
End
No
Return null

Figure 15: The workflow of the `Find` method in the `OperationEnumerator` class. The internal buffer threshold is half the buffer size, which is two times the number of physical cores in the computer.

None of the abovementioned producers have internal status to keep track of intermediate results and none of them have a specialized operation type.

Since there is no particular order of processing of the individual tiles in the input data, the `AbstractOperationGenerator` in each producer just iterates from one end of the output matrix to the other.

The `IsRunnable` method in each producer merely checks to see if the ingoing tile is ready – except for `Multiply` and `PlusMultiply` which checks the entire row and column required.

3.3.5 The LUFactorization producer

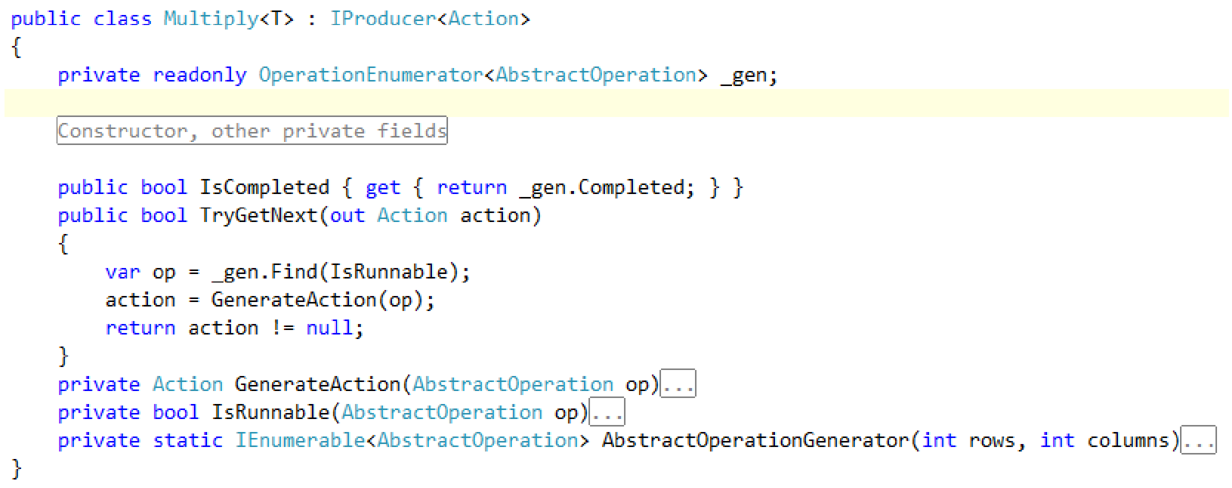
The `LUFactorization` producer is a direct implementation of the algorithm described in [Ske09]. The `AbstractOperationGenerator` produces the suggested optimal computation sequence in section 3.2.1 in [Ske09]. See listing 1 on the following page for our implementation of the `AbstractOperationGenerator` method.

Each `AbstractOperation` returned contains an operation type, which can be an LU, L, U, or A, corresponding to lines 1 and 12, 4, 5, and 9 in the pseudocode in figure 3 on page 12 respectively.

The producer uses an internal status table to keep track of progress. Each position in the status table contains the time step from figure 3 on page 12, with the value of -1 indicating the completion of the corresponding tile. The status table is used by the `IsRunnable` method to verify that conditions are met before allowing a computational

```
private static IEnumerable<AbstractOperation<OpType>> AbstractOperationGenerator(int N)
{
  // PDS elimination algorithm from Stigs article §3.2.1
  for (int stage = 1, endStage = 3 * (N - 1) + 1; stage <= endStage; stage++)
  {
    // Lower bound: The first sweep completes after $2*N-1$ stages and at each succesive stage another
    //   sweep completes. Thus at stage $S > 2*N-1, S-(2*N-1)$ sweeps have completed and $S-(2*N-1)+1$
    //   is the first to be processed.
      // Upper bound: For every third _completed_ stages, a new sweep can start. Thus at stage four
      //   the second sweep can start.
    for (int sweep = System.Math.Max(1, stage - (2 * N - 1) + 1), endSweep = (stage - 1) / 3 + 1; sweep <= endSweep;
        sweep++)
    {
      // generate sweep
      // sweep = is the diagonal sweep number, $sweep \in [1, N]$
      // tsum  = is the sum of the indices in the antidiagonal line to process, $tsum \in [2*sweep, 2*N]$
      // tsum is calculated like this: the index sum of elements being processed at stage $S$
      //   is $S+1$. Each sweep is one step behind the previous, and thus is at index sum
      //   $stage+1-(sweep-1)$. (Sweeps start at one).
      int tsum = stage + 1 - (sweep - 1);
      int iMax = System.Math.Min(N, tsum - sweep);
      int iMin = System.Math.Max(tsum - N, sweep);

      // if it has a diagonal element, do it first
      if (tsum % 2 == 0)
      {
        var i = tsum / 2;

        if (i == sweep)
        {
          yield return new AbstractOperation<OpType>(i, i, OpType.LU);
          continue; // jump to start of for loop again
        }
        yield return new AbstractOperation<OpType>(i, i, sweep, OpType.A);
      }

      if (tsum - sweep <= N)
      {
        // do $L_{ij}$ and $U_{ij}$ second
        yield return new AbstractOperation<OpType>(iMax, sweep, OpType.L);
        yield return new AbstractOperation<OpType>(sweep, tsum - iMin, OpType.U);
      }

      // walk the anti diagonal with indices $tsum-j, j$
      // The j index of the sub matrix to be updated using
      // $A^{(k)}_{i,j}$ operations is bounded by $sweep+1$ and $tsum-(sweep+1)$
      // when above the antidiagonal and $tsum-N$ and $N$ when below
      // the antidiagonal.
      for (int j = System.Math.Max(sweep + 1, tsum - N); j <= System.Math.Min(tsum - (sweep + 1), N); j++)
      {
        // skip the diagonal, already calculated above
        if (j != tsum - j)
        {
          yield return new AbstractOperation<OpType>(tsum - j, j, sweep, OpType.A);
        }
      }
    }
  }
}
```

Listing 1: This listing shows the implementation of the modified PDS algorithm from [Ske09]. Take special note of the "`yield return`" statements. The `yield return` statements are part of a .net construct called an iterator block. "When a `yield return` statement is reached, the current location is stored. Execution is restarted from this location the next time that the iterator is called" (as described in the MSDN library). With all the state management handled by .net, creating the generator as specified in [Ske09] became much easier. Mathematical details are described in the code comments.



```
public class Multiply<T> : IProducer<Action>
{
    private readonly OperationEnumerator<AbstractOperation> _gen;

    Constructor, other private fields

    public bool IsCompleted { get { return _gen.Completed; } }
    public bool TryGetNext(out Action action)
    {
        var op = _gen.Find(IsRunnable);
        action = GenerateAction(op);
        return action != null;
    }
    private Action GenerateAction(AbstractOperation op)...
    private bool IsRunnable(AbstractOperation op)...
    private static IEnumerable<AbstractOperation> AbstractOperationGenerator(int rows, int columns)...
}
```

Figure 16: A collapsed version of the parallel matrix multiply operation. All other matrix operations have an identical implementation of `IsCompleted` and `TryGetNext`.

kernel to be processed.

3.3.6 The MinusMatrixInverseMatrixMultiply producer

As with the `LUFactorization` producer, we have managed to implement the pipelined minus – matrix – inverse-matrix multiply algorithm derived in section 2.3.2 on page 13. Listing 2 on page 29 shows our implementation of the scheduling.

Each `AbstractOperation` returned contains an operation type, which can be an A, Bb, Bc, or C, where A corresponds to line 5 in step 1, Bb corresponds to line 3 and 8 in step 1, Bc corresponds to line 5 in step 1 and C corresponds to line 3 and 8 in step 2, from figure 5 on page 14.

To keep track of the progress of the intermediate results, we use two status tables, one for each step. As with LU-factorization, each position in the status table contains the time step, with the value of -1 indicating the completion of the corresponding tile.

Since it is possible to perform the calculations of each row in parallel, the `AbstractOperationGenerator` of `MinusMatrixInverseMatrixMultiply` will not produce `AbstractOperations` directly, but instead return an `AbstractOperationGenerator` for each of the rows in the result matrix. So it is in essence an `AbstractOperationGenerator` generator. To enable pipelining of the rows we use a special version of the `OperationEnumerator` class mentioned earlier, `PipelinedOperationEnumerator`; the `PipelinedOperationEnumerator` class does this by always trying to find a runnable computational kernel in the first row in its pipeline, only looking for runnable computational kernels in the following rows if that fails.

The internal buffer in the `PipelinedOperationEnumerator` class is never longer than the number of physical cores in the computer, since there is no need to have any more ready to keep all processors busy.

3.3.7 The Inverse producer

As noted in 2.3.3 on page 14, the matrix inverse scheduling follow the same pattern as the minus – matrix – inverse-matrix multiply scheduling.

From an implementation standpoint, they are also quite similar. It also uses two internal status tables to keep track of progress. The `AbstractOperationGenerator` of the matrix inverter producer is also an `AbstractOperationGenerator` generator, the only difference is that the `AbstractOperationGenerators` that are generated generates `AbstractOperations` over each column in the result matrix, instead of each row, otherwise it is the same concept.

Each `AbstractOperation` returned contains an operation type, which can be either a F, R, G, or H, where F corresponds to line 3 and line 8 in step 1, R corresponds to line 5 in step 1, G corresponds to line 3 and 8 in step 2 and H corresponds to line 5 in step 2 (see psuedocode in figure 7 on page 15).

The code in listing 3 on page 30 displays the `AbstractOperationGenerators`.

3.3.8 The BlockTridiagonalMatrixInverse producer

To calculate the inverse of a block tridiagonal matrix we use the `BlockTridiagonalMatrixInverse` producer. It differs from the other producers in that it produces other producers – i.e. the previously mentioned matrix operations, and is thus not directly compatible with the `Manager` class.
The `PipelinedBlockTridiagonalMatrixInverse` and `NonPipelinedBlockTridiagonalMatrixInverse` classes are wrappers that acts as an intermediate between the `Manager` class and the `BlockTridiagonalMatrixInverse` producer. As the names suggest, the first tries to pipeline the matrix operations coming from `BlockTridiagonalMatrixInverse`, while the latter does not.

The non pipelined wrapper does not advance to the next matrix operation until the current one is completed. The pipelined wrapper keeps two matrix operations at hand, issuing runnable computational kernels from the secondary only when none are available from the primary. When the primary is completed, the secondary is promoted and a new one is retrieved in its place.

Because of the serial nature of the calculations of formulae (1) through (6), we decided to have only two operations in the pipeline since there will almost never be any runnable computational kernels available in a third matrix operation.

We have chosen to implement a straightforward scheduling for the computation of the formulae from section 2.1 on page 9. The first operation is a tile operation, after that comes the upward sweep followed by the downward sweep and then the final calculations. Once everything is done we untile the block tridiagonal matrix again.

In order to pipeline matrix operations, they need to be created before their input data exists. This poses a challenge since it requires a pointer to the future location of the input data. We solved this with the OperationResult class mentioned in 3.2 on page 20. It can be shared between different matrix operations, one using it to save the result in and one or more using it as input.

The `IsCompleted` property and the bit table of the `OperationResults` class are used to communicate the status of the entire block matrix or individual tiles in the block matrix between matrix operations. There need not be data embedded in an `OperationResult`, it can simply exist as a link between matrix operations until one of them decides to fill in data and update the appropriate status.

3.3.9 Possible improvements and issues

Many of the obvious improvements we see are related to pipelining.

When it comes to memory usage we are quite wasteful, performing almost none of the calculations inplace and keeping the results of the upward and downward sweeps in memory through the entire block tridiagonal matrix inverse computation. If we where to analyse the individual computations closer we would be able to determine when it is safe to throw away unneeded intermediate results, as well as which would be suitable as inplace operations.

Another realm we see possibilities in is how aggressive some of the matrix operations are about starting their calculations. Currently `MinusMatrixInverseMatrixMultiply`, `Inverse`, and `LUFactorize` all wait until their entire input is ready. In theory, they could start computation as soon as the required number of tiles is ready from the previous matrix operation. This is not trivial though, since one matrix operation has to be absolutely sure it does no longer need a tile in its result before flagging it as done. Otherwise, subsequent matrix operations performing inplace operations might alter the data affecting the calculations of the remaining tiles in the previous matrix operation.

When comparing the results of a single threaded tiled block tridiagonal matrix inversion with the results of the one in parallel, the two sometimes differ by around $1.0E^{-10}$. We believe that this is due to unintended reordering of computational kernels. A finer grained tuning of when the different matrix operations assume its input data is ready should solve this inaccuracy.

There is also the already mentioned issue with the race condition in the `GetWork` method (see 3.3.1 on page 21) that might give us a slight speedup in general during parallel operations if fixed.

All of the above issues seem within our grasp, the only thing missing is time.

```
private static IEnumerable<OperationEnumerator<AbstractOperation<OpType>>> AbstractOperationGenerator(int M, int N)
{
  for (int i = 1; i <= M; i++)
  {
    yield return new OperationEnumerator<AbstractOperation<OpType>>(B_RowActionGenerator(i, N),
        Constants.MAX_QUEUE_LENGTH);
  }

  for (int i = 1; i <= M; i++)
  {
    yield return new OperationEnumerator<AbstractOperation<OpType>>(C_RowActionGenerator(i, N),
        Constants.MAX_QUEUE_LENGTH);
  }
}

// Implementation of PIM algorithm from [Ske09], figure 7.
private static IEnumerable<AbstractOperation<OpType>> B_RowActionGenerator(int i, int N)
{
  for (int step = 1; step <= 2 * (N - 1) + 1; step++)
  {
    // The first N steps, sweep 1 is the first.
    // From then on one sweep is completed at every step.
    int sweep = System.Math.Max(0, step - N);
    for (int j = System.Math.Min(step, N); j >= step / 2 + 1; j--)
    {
      sweep++;
      if (j == sweep)
      {
        // First operation is B_ij
        yield return new AbstractOperation<OpType>(i, j, j, OpType.Bb);
      }
      else
      {
        // The rest are A_ij^(sweep) operations
        yield return new AbstractOperation<OpType>(i, j, sweep, OpType.A);
      }
    }
  }
}

// A PIM based on [Ske09] figure 7, but moving right to left.
// See the minus matrix inverse matrix multiply scheduling figure elsewhere in this paper.
private static IEnumerable<AbstractOperation<OpType>> C_RowActionGenerator(int i, int N)
{
  for (int step = 1; step <= 2 * (N - 1) + 1; step++)
  {
    int sweep = System.Math.Max(0, step - N);

    // As seen in [Ske09] figure 7, the first N steps the j
    // index start at step, the rest of the steps j starts
    // at N. It then counts down to, as seen in the figure, sweep/2+1.
    for (int j = System.Math.Max(N - (step - 1), 1); j <= N - (step / 2); j++)
    {
      sweep++;
      if (j == N - (sweep - 1))
      {
        yield return new AbstractOperation<OpType>(i, j, sweep, OpType.C);
      }
      else
      {
        yield return new AbstractOperation<OpType>(i, j, sweep, OpType.Bc);
      }
    }
  }
}
```

Listing 2: This listing shows the implementation of the minus – matrix – inverse-matrix multiply scheduling algorithm.

```
private static IEnumerable<OperationEnumerator<AbstractOperation<OpType>>> AbstractOperationGenerator(int N)
{
  for (int i = 1; i <= N; i++)
  {
    yield return new OperationEnumerator<AbstractOperation<OpType>>(F_ColumnAbstractActionGenerator(i, N),
        Constants.MAX_QUEUE_LENGTH);
  }

  for (int i = 1; i <= N; i++)
  {
    yield return new OperationEnumerator<AbstractOperation<OpType>>(G_ColumnAbstractActionGenerator(i, N),
        Constants.MAX_QUEUE_LENGTH);
  }
}

// A generator for the schedule of Inverse step 1. It is the
// same schedule as that of minus matrix inverse matrix multiply, except
// that i and j are interchanged
// and some elements are skipped as they do not contribute to the result.
private static IEnumerable<AbstractOperation<OpType>> F_ColumnAbstractActionGenerator(int j, int N)
{
  // Sweeps starting at elements above the diagonal are skipped
  // by starting at the step where sweep j starts.
  for (int step = 2 * (j - 1) + 1; step <= 2 * (N - 1) + 1; step++)
  {
    // Skip to sweep j
    int sweep = System.Math.Max(j - 1, step - N);

    // For the N steps, j - 1 rows
    // are skipped.
    for (int i = System.Math.Min(step - (j - 1), N); i >= step / 2 + 1; i--)
    {
      sweep++;
      if (i == sweep)
      {
        // The timestep (third parameter) is moved to begin at 1.
        yield return new AbstractOperation<OpType>(i, j, sweep - (j - 1), OpType.F);
      }
      else
      {
        // The timestep (third parameter) is moved to begin at 1.
        yield return new AbstractOperation<OpType>(i, j, sweep - (j - 1) - 1, OpType.R);
      }
    }
  }
}

// A generator for the schedule of Inverse step 2. It is identical to the scheduler
// of minus matrix inverse matrix multiply step 2, with i and j
// interchanged.
private static IEnumerable<AbstractOperation<OpType>> G_ColumnAbstractActionGenerator(int j, int N)
{
  for (int step = 1; step <= 2 * (N - 1) + 1; step++)
  {
    int sweep = System.Math.Max(0, step - N);
    for (int i = System.Math.Max(N - (step - 1), 1); i <= N - (step / 2); i++)
    {
      sweep++;
      if (i == N - (sweep - 1))
      {
        yield return new AbstractOperation<OpType>(i, j, sweep, OpType.G);
      }
      else
      {
        yield return new AbstractOperation<OpType>(i, j, sweep - 1, OpType.H);
      }
    }
  }
}
```

Listing 3: This listing shows the implementation of the block matrix inverse scheduling algortihm.

4 Performance measurements

In this section we will present our performance measurements, the results they produced and a discussion of these. To keep the main report within a tolerable length, we try to limit the number of tables and graphs, and refer to the appendix as well as the online copy or the attached media for all our test results. Only results deemed important are included, but most if not all are discussed.

4.1 Experiment setup

During the first months of development we only had access to a laptop with an Intel Core Due 2 processor, limiting our testing ability. Later in the process we gained access to an eight core machine due to Brian Vinter and the Minimum intrusion Grid (MiG)[2] team at DIKU, which we are very grateful for. All our results presented in this section are based on experiments performed on this computer.

Hardware (DIKUs MiG eight core system):

- 2 x Intel Xeon Processor E5310 @ 1.6 GHz, with 8 MB Level 2 cache, 128 KB Level 1 cache
- 8 GB RAM
- Running Linux 2.6.28-14-server Ubuntu SMP
- Mono version 2.0.1

Our software platform is the Microsoft.NET[3] platform, but we are able run our code without too many problems on Linux through the Mono[4] runtime. We did run into a few issues with the version of Mono installed in DIKUs system; specifically, the garbage collector would crash when we tried to perform measurements with very large datasets. The issue is fixed in a later version of Mono but unfortunately, it was not possible to install that on the test computer.

Another limitation with our setup is that we were physically unable to disable a certain number of cores. We desired to run our measurements with 1, 2, 4, 6, and 8 threads active, and had we been able to disable for example four of the cores when running measurements with four threads, the impact of other processes needing CPU time would be equalled out. With that said, we do not consider this a big issue but it is noteworthy nonetheless.

Table 1 on the following page lists all the experiments we performed. All experiments were conducted in both single threaded, single threaded tiled and parallel modes. For

[2]www.migrid.org

[3]www.microsoft.com/NET/

[4]www.mono-project.com - An open source, cross-platform, implementation of C# and the CLR that is binary compatible with Microsoft.NET.

the parallel modes, all measurements were run with 1, 2, 4, 6, and 8 threads. For both single threaded tiled and parallel mode the tile size used ranged from 10 to 150. Some experiments were not feasible or meaningful with some tile sizes, either because the running time became too short or because Mono would crash when a small tile size resulted in too many tiles, i.e. objects, in memory.

We use the following notation in this section:

- BTM is short for Block Tridiagonal Matrix
- 100x150x250 means BTM with 100 elements along the diagonal, giving a total of 298 $(3 * 100 - 2)$ elements. Each element is a block matrix of a random size between 150 and 250 containing random numbers.
- 5000x5000 means a matrix of 5000 by 5000 elements (numbers), all random values.

Operation	**BTM**	**Matrix**	**Size**
LU-factorization	–	√	3000x3000
Inverse	–	√	2500x2500
Minus – matrix – inverse-matrix multiply	–	√	2500x2500
PlusMultiply	–	√	2500x2500
Multiply	–	√	2500x2500
MinusPlusPlus	–	√	5000x5000
Block Tridiagonal Matrix Inverse	√	–	50x100x200
Block Tridiagonal Matrix Inverse	√	–	200x100x200
Block Tridiagonal Matrix Inverse	√	–	100x50x100
Block Tridiagonal Matrix Inverse	√	–	100x100x200
Block Tridiagonal Matrix Inverse	√	–	100x150x250
Block Tridiagonal Matrix Inverse	√	–	10x500x500
Block Tridiagonal Matrix Inverse	√	–	10x750x750

Table 1: Overview of experiments and matrix sizes used.

There is an obvious upper bound on how large we can make in particular the block tridiagonal matrices for testing due to the previously mentioned Mono bug, for example, a block tridiagonal matrix of 100x300x500 would not run. Instead, we reduced the number of block matrices, allowing us to increase the block size.

We were also unable to run the individual matrix operations with larger datasets than those specified. Smaller datasets resulted in very short running times, which did not yield any conclusive results. Nonetheless the chosen sizes are within the magnitude set forth in [Ske09].

Statistics gathered during measurements The statistics gathered during our measurements include the following:

- The total running time of the operation
- The number of threads used (parallel mode only)
- The tile size used (single threaded tiled and parallel mode only)
- The total number of computational kernels processed (parallel mode only)
- The total number of times workers failed to retrieve a runnable computational kernel from the producer (parallel mode only)
- The total number of times workers received a runnable computational kernel from the secondary producer (only in parallel pipelined mode – i.e. block tridiagonal matrix inversion).

Besides allowing us to calculate speedups, we also use the following formulae:

$$Speedup = \frac{running\ time\ of\ baseline\ implementation}{running\ time\ of\ experiment}$$

$$\%\ times\ workers\ did\ not\ receive\ work = \frac{\#times\ failed\ to\ get\ work}{(\#times\ failed\ to\ get\ work\ +\ \#computational\ kernels)}$$

$$\%\ computational\ kernels\ pipelined = \frac{\#times\ received\ work\ from\ secondary\ producer}{\#computational\ kernels}$$

For two of the block tridiagonal matrix inversion experiments, we also logged:

- The total running time in ticks
- The total number of ticks workers slept, waiting for another worker to notify them of possible new available computational kernels.
- The total number of ticks workers spent spinning, waiting to be allowed to request a computational kernel (the spin wait lock around the TryGetWork method in the producer).

The above statistics allowed us to calculate percentage of time spent sleeping as well as the percentage of time spent contending the spin wait lock. These are calculated like this:

$$\%\ spinning\ time = \frac{\#ticks\ spinning}{\#threads \times running\ time\ in\ ticks}$$

$$\%\ sleeping\ time = \frac{\#sleeping\ ticks}{\#threads\ \times\ running\ time\ in\ ticks}$$

Since the total running time is not based on how many ticks the program actually used, but rather the amount of ticks that passed during the entire execution of the experiment, the above formulae present the best case scenario, where none of the threads are interrupted by the operating system or other components in the runtime like the garbage collector. Nevertheless, we feel they are still valid and bring good insight into the overhead caused by our scheduling algorithms.

4.2 Experiment results and analysis

In table 2 we present the best possible speedups achieved in the different matrix operations, as well as the configuration used. In the following subsections we will go into the result from the experiments for each matrix operation.

Matrix operation	**Tile size**	**Matrix or BTM size**	**Best speedup**
LU-factorization	100	3000x3000	7.734
Inverse	90	2500x2500	7.742
Minus – matrix – inverse-matrix multiply	90	2500x2500	7.772
PlusMultiply	150	2500x2500	7.838
Multiply	150	2500x2500	7.858
MinsPlusPlus	10	5000x5000	2.716
Block Tridiagonal Matrix Inverse	80	10x750x750	7.348

Table 2: Overview of best achieved speedups using 8 threads. It turned out to be very hard to achieve speedups with a larger number of threads in the MinusPlusPlus operatoin. The speedup in the table was logged with only two threads active. We discuss the issues with MinusPlusPlus in section 4.2.3 on page 39

For a baseline comparison, we used the results collected from the single threaded and single threaded tiled implementations. Interestingly, going from a non-tiled to a tiled implementation yields significant speedups. The table 3 on the following page shows the initial speedups we attain going from non-tiled to tiled.

The reason for the speedups we get when tiling may be that the processor keeps the individual tiles in local cache longer while performing calculations, limiting the overhead of communication between main memory and the CPU, resulting in better utilization of the processor caches.

Matrix operation	**Tile size**	**Matrix or BTM size**	**Speedup**
LU-factorization	50	3000x3000	2.110
Inverse	40	2500x2500	4.329
Minus – matrix – inverse-matrix multiply	50	2500x2500	5.085
PlusMultiply	60	2500x2500	2.776
Multiply	60	2500x2500	2.793
MinusPlusPlus	100	5000x5000	5.065
Block Tridiagonal Matrix Inverse	40	10x750x750	2.368

Table 3: Single threaded to single threaded tiled speedups

Tiling is not always the best solution though. On DIKUs computer, where the processor has a rather large L1 cache available, the added overhead of tiling results in slowdowns when the blocks in a block tridiagonal matrix gets below approximately 150x150.

4.2.1 Presentation and analysis of LU-factorization

Looking at the chart in figure 17 on the next page we see an almost near perfect speedup for the larger tile sizes. The speedup does loose a little ground when the thread count go up, but we suspect speedups to continue to look decent with even more cores available as well.

Tiles size\Threads	**1**	**2**	**4**	**6**	**8**
10	0.760	1.055	1.418	1.606	1.590
20	0.912	1.585	2.634	3.426	3.914
30	0.946	1.777	3.261	4.530	5.596
40	0.943	1.783	3.117	4.637	4.878
50	0.961	1.877	3.592	4.966	6.601
60	0.982	1.936	3.781	5.321	7.179
70	0.985	1.947	3.850	5.515	7.444
80	0.990	1.967	3.904	5.595	7.619
90	0.989	1.970	3.916	5.672	**7.647**
100	0.993	1.981	3.932	5.610	7.734
150	0.993	1.982	3.944	5.854	7.722

Table 4: The speedup table for parallel LU-factorization. This table is charted in figure 17 on the next page.

The best case in this experiment, a 3000x3000 matrix tiled with tile size 100, took around 9.855 seconds to complete, and during that time, workers were turned down less than 1% of the times new work was requested – i.e. when there were no runnable computational kernels ready. While we did not measure the actual time wasted when a worker did

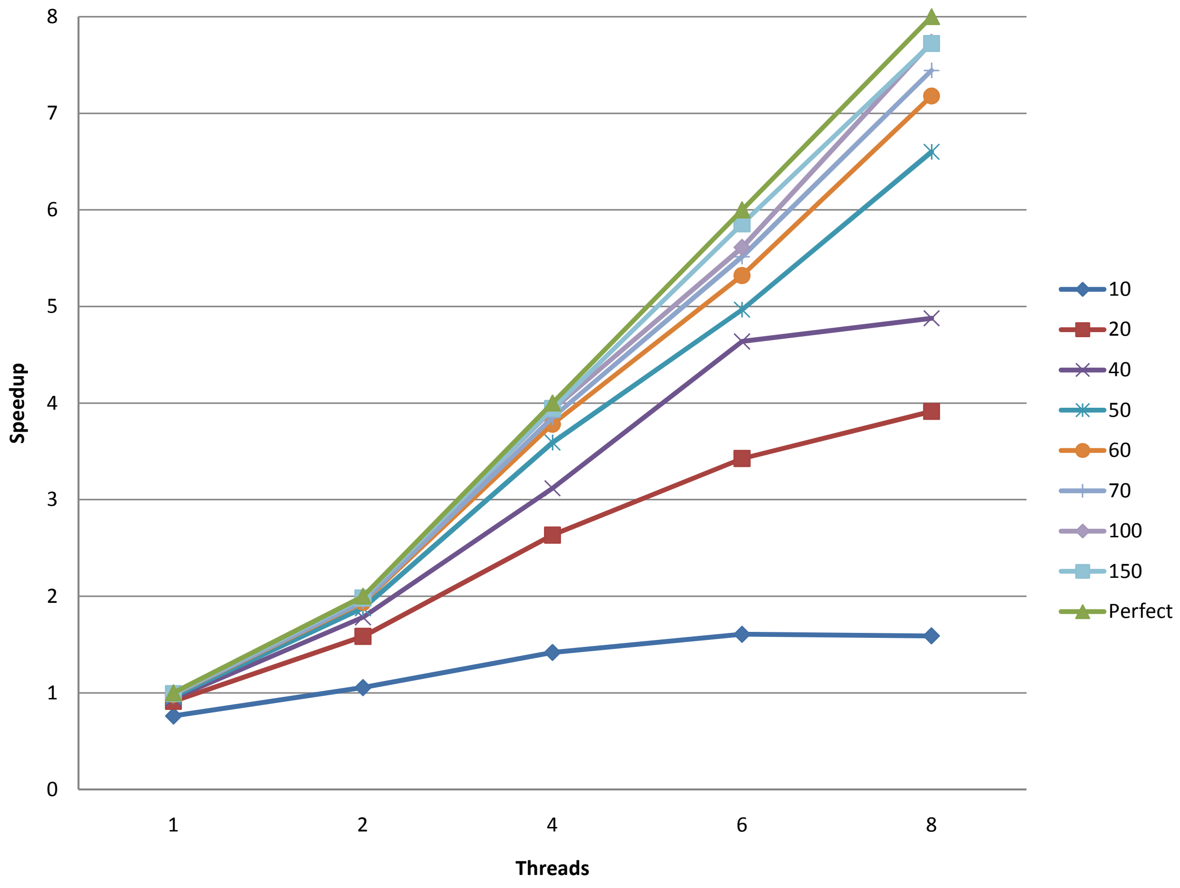


Figure 17: Speedup chart for parallel LU-factorization. Note that some tile sizes are filtered out to make the chart more readable, the full dataset can be found in table 4 on the previous page

not receive work from the LU-factorize producer, the very low number of "misses" gives us a strong indication that the modified PDS algorithm presented in [Ske09] and reiterated in section 2.3.1 on page 11 does indeed work well.

We clearly see that the smaller tile sizes are not able to produce a respectable speedup. Even though workers were almost never turned down asking for work (1.22E-6% of the time for tile size 10, or in numbers, 11 times out of 9,045,061), the large amount of small tiles resulted in very fast computational kernels and a lot more scheduling. The smaller tile sizes also resulted in poor utilization of processor cache, by not coming close to filling the cache, which certainly did not help running times either.

With larger datasets it is possible that we could have produced slightly better speedups. With smaller datasets we saw speedups drop when the matrix size came close to the tile size, for example in the case of a 150x150 block matrix. This is expected as the overhead of scheduling takes a more significant amount of time and there are fewer tiles than workers.

Running times and other statistics are printed in the appendix in table 23 on page 62.

4.2.2 Presentation and analysis of Inverse and Minus – Matrix – Inverse-Matrix Multiply

Our experiments with Inverse and Minus – Matrix – Inverse-Matrix Multiply yielded very similar results (and oddities), so we will look at them together. Considering the similar nature of their implementation, this is no surprise.

In general, tile sizes in the range of 70 to 150 all resulted in speedups above 7.4, and speedups are, as with LU-factorization, close to perfect with those tile sizes.

As the charts in figure 19 on page 39 and figure 18 reveal, we calculated the speedup based on the results of our parallel implementation running with one thread. We did this because our parallel implementation actually produced better running times with just one thread than the single threaded tiled version.

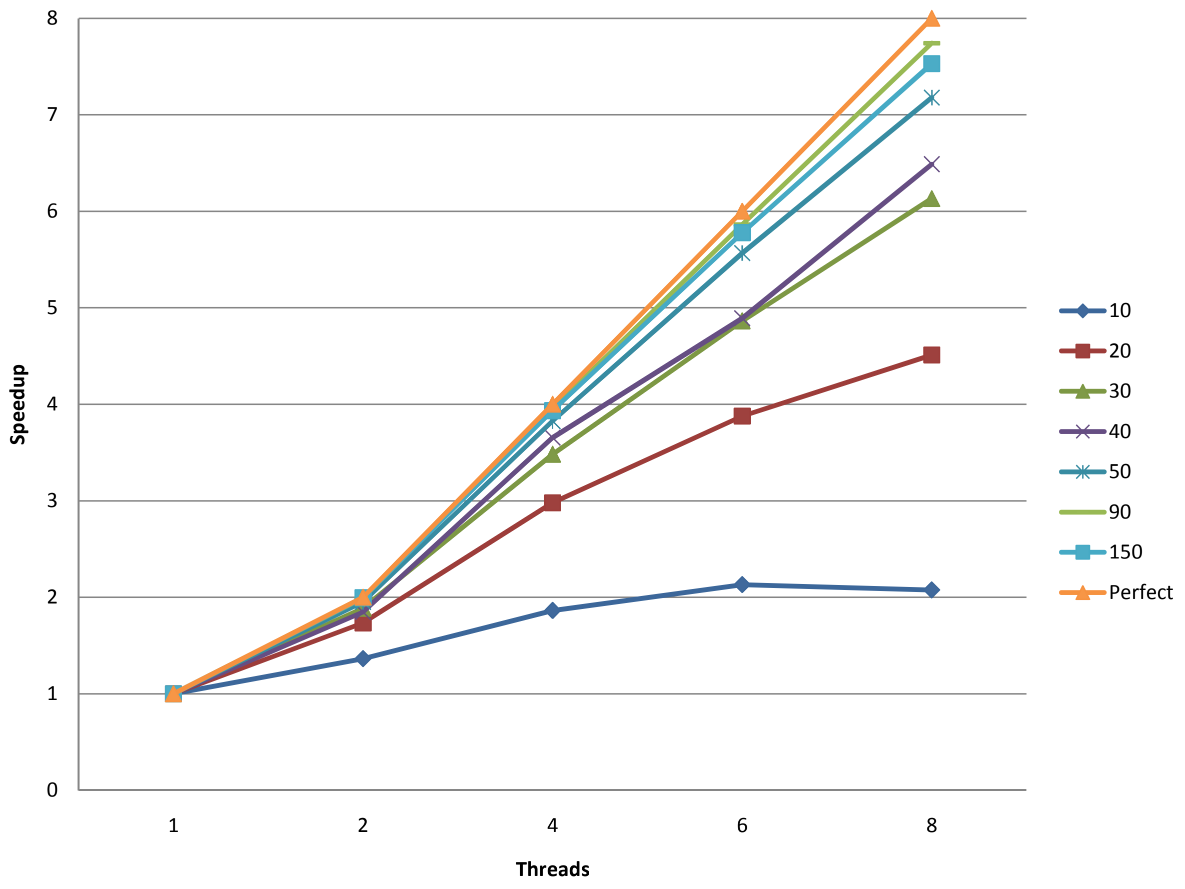


Figure 18: Speedup chart for parallel Inverse. Note that some tile sizes are filtered out to make the chart more readable, the full dataset can be found in table 5 on the next page

The speedup from the single threaded tiled version to the parallel version using one thread is 1.4 on average over all tile sizes, and comparing the single threaded tiled version to the parallel version running with eight threads yielded speedups above 10.0, in the case

Tile size\Threads	1	2	4	6	8
10	1.000	1.364	1.863	2.131	2.076
20	1.000	1.733	2.979	3.877	4.511
30	1.000	1.886	3.481	4.864	6.131
40	1.000	1.847	3.654	4.891	6.488
50	1.000	1.952	3.827	5.568	7.178
60	1.000	1.973	3.859	5.679	7.420
70	1.000	1.992	3.933	5.800	7.606
80	1.000	1.992	3.945	5.830	7.707
90	1.000	1.990	3.954	5.863	**7.742**
100	1.000	1.993	3.961	5.868	7.630
150	1.000	1.995	3.935	5.779	7.531

Table 5: The speedup table for parallel Inverse. This table is charted in figure 18 on the previous page.

Tile size\Threads	1	2	4	6	8
10	1.000	1.429	1.878	2.131	2.090
20	1.000	1.716	2.899	3.777	4.389
30	1.000	1.893	3.491	4.871	6.023
40	1.000	1.943	3.358	4.738	5.661
50	1.000	1.984	3.780	5.585	7.067
60	1.000	1.988	3.870	5.673	7.398
70	1.000	1.994	3.934	5.824	7.642
80	1.000	1.993	3.954	5.863	7.718
90	1.000	1.995	3.967	5.887	**7.772**
100	1.000	1.998	3.974	5.905	7.733
150	1.000	1.994	3.949	5.880	7.667

Table 6: The speedup table for parallel Minus – Matrix – Inverse-Matrix Multiply. This table is charted in figure 19 on the next page.

of Inverse, and above 11.5 with Minus – Matrix – Inverse-Matrix Multiply.

One possible explanation could be that the scheduling implemented in the parallel version has an effect even with only one thread in play. The order in which the different tiles are computed may result in less communication between processor and memory, resulting in better speed. Also, very little synchronisation is required when only one thread is running.

Another reason could be a less than optimal single threaded tiled implementation of Inverse and Minus – Matrix – Inverse-Matrix Multiply on our part, even though we produce significant speedups compared to our non-tiled implementation.

Oddities aside, we do think that the PIM algorithm presented in [Ske09] has proven itself, at least with our experiment setup. With the tile sizes that resulted in the best

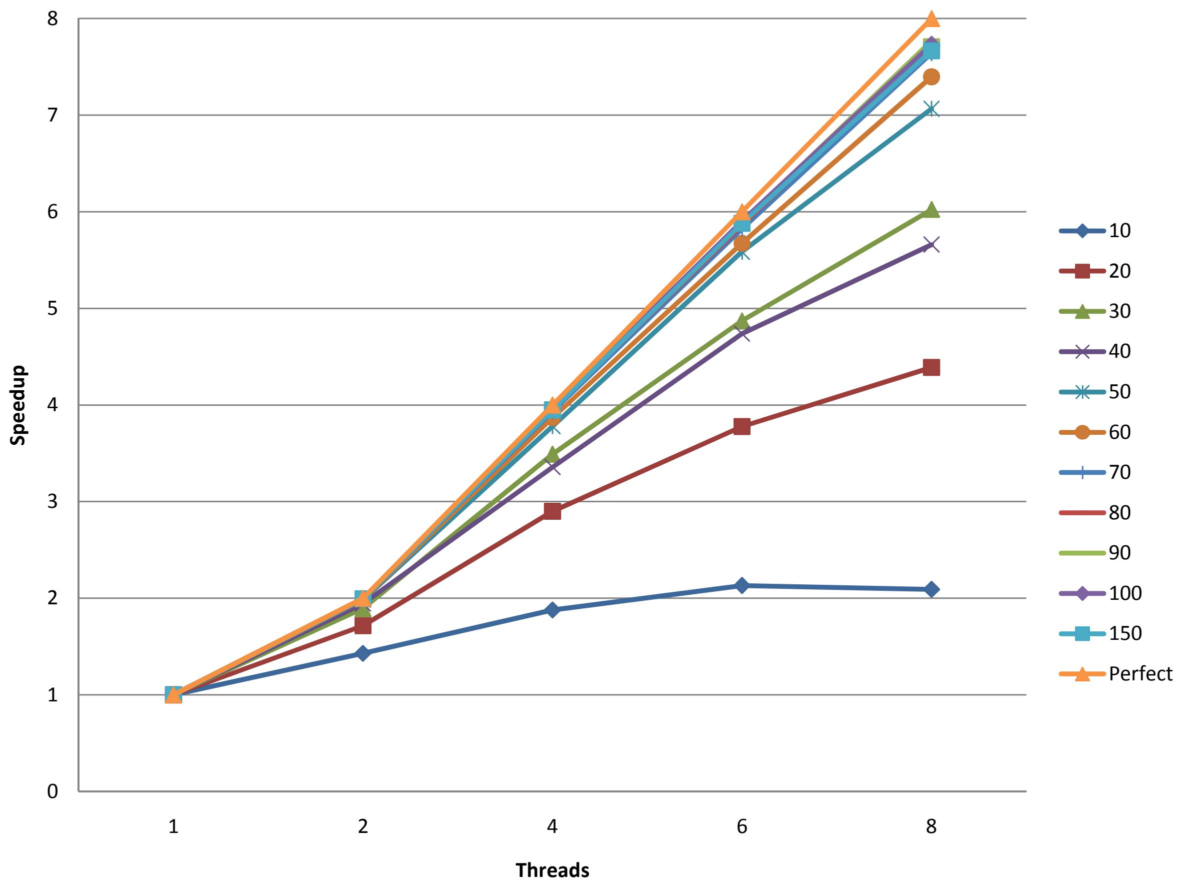


Figure 19: Speedup chart for parallel Minus – Matrix – Inverse-Matrix Multiply. Note that some tile sizes are filtered out to make the chart more readable, the full dataset can be found in table 6 on the previous page

speedups, the number of times a worker did not get a runnable computational kernel from the producer was less than 2% in both operations, with a total running time at around 11 seconds for Inverse and 16 seconds for Minus – Matrix – Inverse-Matrix Multiply.

Running times and other statistics are printed in the appendix in table 24 on page 63 and in table 25 on page 64.

4.2.3 Presentation and analysis of Multiply, Plus Multiply, Minus Plus Plus

Both Multiply and Plus Multiply have near optimal speedups of about 7.8, which is no big surprise since they are both delightfully parallel. We note that both keep all threads busy at all time; there is no waiting for runnable computational kernels at any time during execution. See table 27 on page 66 and table 28 on page 67 for running times and other statistics.

We also expected great speedups for our Minus Plus Plus operation, but it turns out that running it in parallel did not produce good results at all. 2.716 was best speedup achieved with a tile size of 10 using 6 threads. Actually, anything above two threads results in disappointing speedups; it was especially bad with eight threads where we actually managed to get a slowdown. Table 26 on page 65 reveals the dreadful results.

The obvious reason lies in the fact that the running time of computational kernels to scheduling overhead ratio is too small, due to the simple nature of the minus and plus matrix operations. We might have been able to get better measurement results if we had been able to run with a larger dataset; this was not possible due to Mono crashing when a large number of objects are created in memory.

In practice, we suspect that one would need a very large matrix – say in the range of 15000x15000 to see a significant speedup with more than a few threads. A matrix of 5000x5000 used in our experiments only took our single threaded tiled implementation 0.9 seconds to complete, not leaving much room for parallelization. In cases with block sizes in the lower thousands, as are the practical use cases reported in [Ske09], one might consider not running MinusPlusPlus in parallel at all.

4.2.4 Presentation and analysis of Block Tridiagonal Matrix Inversion

For the block tridiagonal matrix inversion experiments we varied the number of blocks as well as their size. The best speedup with eight threads in each experiment is displayed in table 7.

BTM size	**Tile size**	**Speedup**
100x50x100	30	1.847
50x100x200	40	4.258
100x100x200	35	4.545
200x100x200	35	4.604
100x150x250	40	5.571
10x500x500	90	7.174
10x750x750	80	7.348

Table 7: Summary of best speedups attained with eight threads in the different experiments conducted.

Looking at the summary table, we clearly see a pattern. Block sizes have great impact on the speedup, while the number of blocks has negligible impact.

When running with only 4 threads, we see decent speedups of around 3.1 on lower block sizes as well. The chart in figure 20 on the following page illustrates this. It shows the best possible speedup achieved for all BTM sizes as a function of thread count. All but 100x50x100 is able to produce a speedup of over 3.0 with 4 threads. Raising the thread count, only BTMs with blocks bigger than 500x500 are able to retain the growth of their speedup.

This is not a surprising result considering the small amount of tiles produced when tiling the smaller block matrices. Tiling a block matrix of size 150x150 with a tile size of 30 only results in 5 blocks, making it hard to keep more than 4 workers busy.

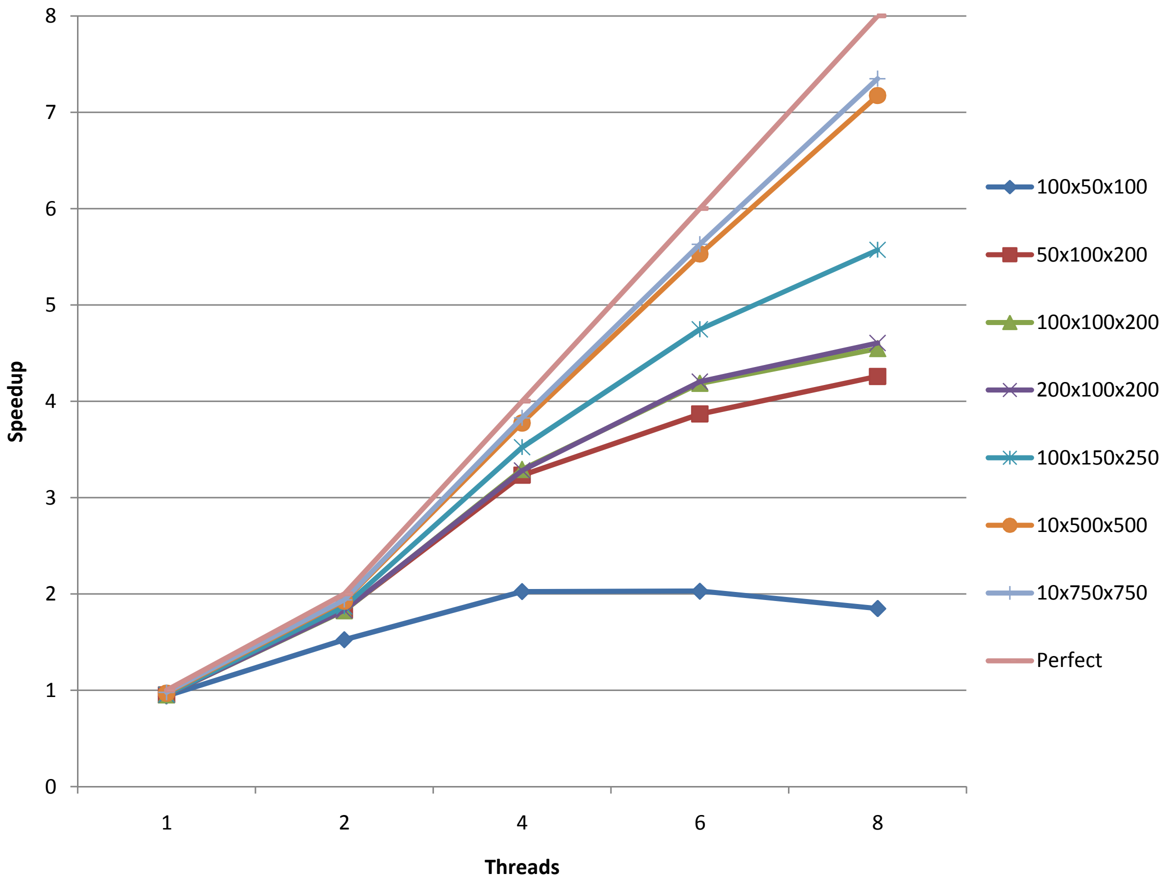


Figure 20: Speedup chart for parallel Block Tridiagonal Matrix Inversion.

If we take a closer look at the numbers gathered in our best case scenario, we processed a 10x750x750 BTM tiled with a tile size of 80, in 35.676 seconds resulting in a speedup of 7.348. That tile size resulted in 10 tiles per block, which in turn resulted in workers getting turned down for runnable computational kernels 9.492% of the time they asked. However, only 3% of the total running time was actually spent sleeping – waiting for other workers to notify them of possible new work being available – and only 0.5% of the running time was spent spinning while waiting their turn to ask the producer for work.

The statistics collected running with a BTM of size 100x100x200, tiled with tile size 35, explains the poor speedup of 4.545. With an average of 5 tiles in each block, the eight threads did not receive runnable computational kernels 31.16% of the time they requested it and spent 11.06% of their time sleeping. Those numbers are not surprising since there are eight workers contending for five tiles in each block. We also registered a large 15.03% of time spent spin waiting for the producer lock, which makes sense since more workers will arrive at the spin wait lock at the same time due to being woken up by workers completing work.

Full tables with all the cited data can be found in appendix A on page 46.

4.3 Possible improvements

While we are happy with the speedups we are able to produce for block tridiagonal matrix inversion, we do see a few obvious areas where we could improve things.

As already mentioned, the attempt at parallelizing the minus plus plus operation should be reconsidered. It speaks in its favour that it is very pipeline friendly; it can start computation on the individual tiles in its input as soon as they are ready, and as soon as a tile in a its result matrix is finished, a following matrix operation can start its work on that particular tile.

In contrast LU-factorization, Inverse and Minus – Matrix – Inverse-Matrix Multiply all lack proper pipelining capabilities. In their current state they have to wait until all their input is completely ready before beginning their computations. The reason for this is that they all clone their input and perform the calculations inplace on the clone. We believe we can alleviate this without changing the general design of our framework given more time.

Besides making some of the matrix operations more pipeline friendly, we could look into breaking apart the serial nature of the tridiagonal block inversion algorithm by interleaving the operations that do not depend on each other. For example, the upward and downward sweeps could be interleaved as they are completely independent.

In general we have observed that BTM operations are not pipelined very much. We expect both enhancements mentioned above to improve on this. In particular we expect avoiding operation interdependencies to give good results. BTMs with smaller block sizes – i.e. few tiles – will benefit the most from improved pipelining.

Another improvement that all matrix operations probably could benefit from, is to completely remove the sleeping state a worker can enter if it does not receive any work from the producer; instead of going to sleep it simply retries until receiving a computational kernel, or the producer completes. This would eliminate the notification step after the processing of a computational kernel and get rid of the race condition mentioned in section 3.3.1 on page 21.

Continuing on that path, designing a manager without a lock around the call to the TryGetWork method could be an adventurous undertaking. The benefits of this might be more significant if the amount of workers (i.e. cores) goes from eight to 16, 32 or even higher. A solution based on a lock free implementation of a linked list or priority queue could be an option. We toyed with a few designs during development, but optimizations performed by the .net compiler and CPU at runtime, like reordering of read and write instructions in some conditions, makes it hard at best to write lock free code, and even more difficult to verify.

4.4 What could have been

Our measurements are far from comprehensive; here are some of the things we would have liked to measure had time allowed:

- Measure the total time spent processing computational kernels (in ticks).
- Measure the total time spent retrieving computational kernels from the producer (this would allow us to better calculate the overhead of the scheduling).
- Measure the actual CPU utilization during execution.
- Measure the cache hit-miss ratio during execution.

With the above data, it would have been possible to get a better idea of how much time was spent during execution by other things like garbage collection, other processes outside our program or the OS itself.

We are, however, uncertain if measuring CPU utilization, cache hit-miss ratio, and advanced profiling in general is possible on the MiG system.

It would also have been great to run all the parallel experiments with the extra ticks statistics we only had time to gather for a few selected BTM experiments. Obviously, a wider range of experiments, especially larger block sizes and higher tile sizes, would have been interesting.

While we do not expect there to be any noteworthy overhead of pipelining individual blocks in a BTM, it would have been good to confirm this. As mentioned in section 4.1 on page 31, the issues with Mono prevented us from playing much with number of blocks while using larger block sizes, so even though DIKUs computer has enough memory to allow us to compute a BTM of 50x750x750 or even 100x750x750, Mono would not play nice.

On the hardware side, it would have been of great interest to run our performance measurements on different hardware, in particular hardware with more than eight cores.

5 Conclusion

In this project we had a chance to gain valuable practical experience with many facets of parallel programming, which we did not have beforehand. We managed to implement a working prototype of a framework that supports inversion of block tridiagonal matrices in parallel, and produce speedups in the range of those derived by Skelboe in [Ske09].

The table below show our best speedups achieved over the different matrix operations we implemented in this project.

Matrix operation	**Tile size**	**Matrix or BTM size**	**Best speedup**
LU-factorization	100	3000x3000	7.734
Inverse	90	2500x2500	7.742
Minus – matrix – inverse-matrix multiply	90	2500x2500	7.772
PlusMultiply	150	2500x2500	7.838
Multiply	150	2500x2500	7.858
MinsPlusPlus	10	5000x5000	2.716
Block Tridiagonal Matrix Inverse	80	10x750x750	7.348

Table 8: Overview of best achieved speedups. It turned out to be very hard to achieve speedups with a larger number of threads in the MinusPlusPlus operatoin. The speedup in the table was logged with only two threads active. We discuss the issues with MinusPlusPlus in section 4.2.3 on page 39

Even though we are satisfied with both our implementation and the results it produces, we see opportunities for improvement. Among these better support for pipelining mentioned earlier in section 4.4 on the preceding page is expected to yield the biggest improvement.

With more time we would have conducted more extensive performance measurements, nonetheless we feel our conclusions are well founded.

We ended up implementing a basic linear algebraic matrix library specially tailored to our requirements, which was not our intention to begin with; the ones we found available on the Internet did unfortunately not meet our needs. While the prototype framework produced was initially only intended to test the theories set forth by Skelboe, we ended up with a very solid design that can easily be used to solve other mathematical problems in need of parallelization. It is also very easy to add support for different data types like complex numbers. Since our design is modular by nature due to our use of generic programming, it is easy to exchange and extend many parts of the framework.

References

[PSH$^+$08] Dan Erik Petersen, Hans Henrik B. Sørensen, Per Christian Hansen, Stig Skelboe, and Kurt Stokbro. Block tridigonal matrix inversion and fast transmission calculations. Journal of Computational Physics, pages 3174–3190, 2008.

[Ske09] Stig Skelboe. The scheduling of a parallel tiled matrix inversion. 2009.

A Measurement results

In this section we list all the statistics gathered which are mentioned in the report. Both those listed here and even more may be found online at http://github.com/egil/Inversion-of-Block-Tridiagonal-Matrices/tree/master/Rapport/Experiments/ or on the CD accompanying this report.

- BTM 10x750x750 - collected statistics: table 9 on page 48
- BTM 10x750x750 - calculated statistics: table 10 on page 49
- BTM 10x750x750 - measurement including extra statistics - collected statistics part 1: table 11 on page 50
- BTM 10x750x750 - measurement including extra statistics - collected statistics part 2: table 12 on page 51
- BTM 10x750x750 - measurement including extra statistics - calculated statistics: table 13 on page 52
- BTM 10x500x500 - collected statistics: table 14 on page 53
- BTM 10x500x500 - calculated statistics: table 15 on page 54
- BTM 100x100x200 - measurement including extra statistics - collected statistics: table 16 on page 55
- BTM 100x100x200 - measurement including extra statistics - calculated statistics: table 17 on page 56
- BTM 100x100x200 - collected and calculated statistics: table 18 on page 57
- BTM 100x50x100 - collected and calculated statistics: table 19 on page 58
- BTM 100x150x250 - collected and calculated statistics: table 20 on page 59
- BTM 200x100x200 - collected and calculated statistics: table 21 on page 60
- BTM 50x100x200 - collected and calculated statistics: table 22 on page 61
- LU-factorization: 3000x3000 - collected and calculated statistics: table 23 on page 62
- Inverse: 2500x2500 - collected and calculated statistics: table 24 on page 63
- Minus – Matrix – Inverse-Matrix Multiply: 2500x2500 - collected and calculated statistics: table 25 on page 64
- MinusPlusPlus: 5000x5000 - collected and calculated statistics: table 26 on page 65

- Multiply: 2500x2500 - collected and calculated statistics: table 27 on page 66
- PlusMultiply: 2500x2500 - collected and calculated statistics: table 28 on page 67

Block Tridiagonal Matrix Inverse: 10x750x750

Running times

Single threaded 09:13.808

Single threaded tiled	**05:57.717**	**04:14.830**	**04:02.341**	**03:55.458**	**03:54.988**	**03:53.867**	**03:55.218**	**04:00.758**	**04:14.633**	**04:21.818**	**04:28.988**	**04:32.645**	**04:37.451**	**04:47.841**	**04:48.856**	**05:00.050**
Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**150**
1	06:57.703	04:25.868	04:10.481	04:01.881	04:01.134	04:00.114	04:01.157	04:06.864	04:20.728	04:28.479	04:36.542	04:40.084	04:44.637	04:56.395	04:56.802	05:08.432
2	05:13.173	02:28.153	02:13.282	02:05.192	02:04.229	02:03.569	02:02.206	02:04.435	02:11.418	02:15.033	02:19.271	02:20.994	02:23.371	02:29.263	02:29.844	02:36.302
4	04:25.798	01:28.466	01:14.024	01:06.712	01:05.160	01:03.929	01:02.521	01:03.396	01:06.812	01:08.412	01:10.654	01:11.499	01:13.664	01:15.783	01:16.498	01:21.295
6	04:13.924	01:08.693	00:54.759	00:47.423	00:45.627	00:44.390	00:42.838	00:43.187	00:45.524	00:46.516	00:47.931	00:48.492	00:50.241	00:52.382	00:52.939	00:55.832
8	04:26.632	00:59.646	00:45.173	00:37.832	00:36.073	00:34.763	00:33.110	00:33.071	00:34.851	00:35.630	00:36.904	00:37.676	00:39.085	00:40.461	00:40.191	00:46.354

Number of times a thread has waited for work

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**150**
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	87	119	104	110	138	137	117	131	145	135	173	135	124	125	141	121
4	338	932	809	803	1609	1068	923	1216	917	923	900	762	846	682	691	683
6	654	2091	2305	2810	4190	3217	2732	2910	2548	2401	2176	1944	2004	2085	1981	1505
8	939	2713	4259	5413	6556	6008	4913	4890	4710	4428	4000	4399	4239	3883	3286	4920

Number of times a thread has completed a computational kernel

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**150**
1	14839408	1993082	997498	586508	404858	265184	134728	89732	56048	42938	32044	23162	16088	16088	10618	6548
2	14839408	1993082	997498	586508	404858	265184	134728	89732	56048	42938	32044	23162	16088	16088	10618	6548
4	14839408	1993082	997498	586508	404858	265184	134728	89732	56048	42938	32044	23162	16088	16088	10618	6548
6	14839408	1993082	997498	586508	404858	265184	134728	89732	56048	42938	32044	23162	16088	16088	10618	6548
8	14839408	1993082	997498	586508	404858	265184	134728	89732	56048	42938	32044	23162	16088	16088	10618	6548

Number of times a thread has received work from a secondary producer (in pipelined mode)

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**150**
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	72	64	75	64	55	63	48	40	37	41	40	37	36	30	34	28
4	262	324	305	308	298	269	345	265	222	264	276	266	219	259	206	263
6	1223	607	567	585	659	612	626	687	583	618	612	597	557	613	543	583
8	808	1578	2316	1190	1070	1116	1294	1120	1083	1057	1070	1067	1045	1138	978	983

Table 9: BTM 10x750x750 - collected statistics

Block Tridiagonal Matrix Inverse: 10x750x750

Speedup

Threads \ Tile size	10	20	25	30	35	40	50	60	70	80	90	100	110	120	130	150
1	0.856	0.958	0.968	0.973	0.975	0.974	0.975	0.975	0.977	0.975	0.973	0.973	0.975	0.971	0.973	0.973
2	1.142	1.720	1.818	1.881	1.892	1.893	1.925	1.935	1.938	1.939	1.931	1.934	1.935	1.928	1.928	1.920
4	1.346	2.881	3.274	3.529	3.606	3.658	3.762	3.798	3.811	3.827	3.807	3.813	3.766	3.798	3.776	3.691
6	1.409	3.710	4.426	4.965	5.150	5.268	5.491	5.575	5.593	5.629	5.612	5.622	5.522	5.495	5.456	5.374
8	1.342	4.272	5.365	6.224	6.514	6.727	7.104	7.280	7.306	**7.348**	7.289	7.237	7.099	7.114	7.187	6.473

% times a thread asked in vain for work (#waited for work / (#waited for work + # computational kernels))

Threads\#tiles	75	38	30	25	22	19	15	13	11	10	9	8	7	7	6	5
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.001%	0.006%	0.010%	0.019%	0.034%	0.052%	0.087%	0.146%	0.258%	0.313%	0.537%	0.579%	0.765%	0.771%	1.311%	1.814%
4	0.002%	0.047%	0.081%	0.137%	0.396%	0.401%	0.680%	1.337%	1.610%	2.104%	2.732%	3.185%	4.996%	4.067%	6.110%	9.445%
6	0.004%	0.105%	0.231%	0.477%	1.024%	1.199%	1.987%	3.141%	4.348%	5.296%	6.359%	7.743%	11.077%	11.473%	15.723%	18.689%
8	0.006%	0.136%	0.425%	0.914%	1.594%	2.215%	3.518%	5.168%	7.752%	9.348%	11.098%	15.961%	20.854%	19.443%	23.633%	42.902%

% computational kernels pipelined = #received work from secondary producer / #total computational kernels

Threads\#tiles	75	38	30	25	22	19	15	13	11	10	9	8	7	7	6	5
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.000%	0.003%	0.008%	0.011%	0.014%	0.024%	0.036%	0.045%	0.066%	0.095%	0.125%	0.160%	0.224%	0.186%	0.320%	0.428%
4	0.002%	0.016%	0.031%	0.053%	0.074%	0.101%	0.256%	0.295%	0.396%	0.615%	0.861%	1.148%	1.361%	1.610%	1.940%	4.016%
6	0.008%	0.030%	0.057%	0.100%	0.163%	0.231%	0.465%	0.766%	1.040%	1.439%	1.910%	2.577%	3.462%	3.810%	5.114%	8.903%
8	0.005%	0.079%	0.232%	0.203%	0.264%	0.421%	0.960%	1.248%	1.932%	2.462%	3.339%	4.607%	6.496%	7.074%	9.211%	15.012%

Table 10: BTM 10x750x750 - calculated statistics

Block Tridiagonal Matrix Inverse: 10x750x750 - measurement including extra statistics - part 1

Running times

	Single threaded			09:13.808												
Single threaded tiled	**05:57.717**	**04:14.830**	**04:02.341**	**03:55.458**	**03:54.988**	**03:53.867**	**03:55.218**	**04:00.758**	**04:14.633**	**04:21.818**	**04:28.988**	**04:32.645**	**04:37.451**	**04:47.841**	**04:48.856**	**05:00.050**
Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**150**
1	07:04.201	04:27.218	04:10.790	04:01.506	04:01.398	04:00.090	04:01.035	04:06.869	04:20.755	04:28.393	04:36.631	04:40.055	04:44.597	04:56.454	04:56.337	05:07.943
2	05:24.213	02:30.465	02:14.203	02:05.218	02:04.153	02:03.923	02:02.117	02:04.424	02:11.357	02:15.187	02:19.112	02:21.057	02:23.306	02:29.255	02:29.905	02:36.401
4	04:25.923	01:28.921	01:14.096	01:06.788	01:05.187	01:04.073	01:02.493	01:03.414	01:06.850	01:08.544	01:10.701	01:11.495	01:13.674	01:15.797	01:16.449	01:21.233
6	04:13.273	01:08.993	00:54.818	00:47.258	00:45.653	00:44.478	00:42.733	00:43.247	00:45.585	00:46.650	00:47.902	00:48.533	00:50.095	00:52.497	00:52.886	00:56.034
8	04:24.180	00:59.827	00:45.269	00:37.963	00:36.092	00:34.898	00:33.124	00:33.143	00:34.862	00:35.676	00:36.827	00:37.663	00:39.192	00:40.266	00:40.239	00:46.414

Running time in ticks

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**150**
1	4242009118	2672179186	2507899501	2415063414	2413984620	2400898821	2410345402	2468694893	2607550786	2683926701	2766313463	2800545048	2845974125	2964544205	2963372551	3079433439
2	3242125100	1504649034	1342029050	1252182068	1241533437	1239232104	1221167443	1244244696	1313573027	1351870739	1391118487	1410565838	1433060309	1492551454	1499046561	1564010115
4	2659232460	889206660	740957472	667883068	651873991	640725279	624931619	634135446	668496846	685436490	707005756	714946810	736743437	757967588	764494502	812329970
6	2532725517	689927121	548175371	472575114	456525275	444782442	427326275	432470680	455852569	466502670	479023647	485326891	500951463	524965901	528864886	560339161
8	2641804649	598273105	452691953	379629722	360915844	348980748	331237664	331426690	348616626	356760231	368272578	376632564	391921142	402658004	402386507	464141922

Number of times a thread has waited for work

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**150**
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	85	122	103	111	139	135	118	130	144	135	155	137	125	124	145	123
4	334	939	803	812	1647	1060	923	1142	935	993	914	768	847	683	700	676
6	636	2107	2199	2699	3654	3255	2765	2871	2640	2424	2240	1913	1960	2151	1896	1533
8	869	2866	4606	5514	6456	6503	5338	5148	4696	4503	4072	4429	4213	3867	3281	4822

Table 11: BTM 10x750x750 - measurement including extra statistics - collected statistics part 1

Block Tridiagonal Matrix Inverse: 10x750x750 - measurement including extra statistics - part 2

Running times

Number of ticks a thread has waited for work

Threads \ Tile size	10	20	25	30	35	40		60	70	80	90	100	110	120	130	150
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	304101	258042	972279	1422697	1612817	1997475	2025225	3538088	4203045	7095577	9846004	12894185	15678763	15351466	26235682	40282350
4	1124453	2123062	3088697	3682706	7742428	7039673	13483165	18405319	25587755	27583538	35688951	40570512	84361395	52863483	79745855	156380504
6	2617478	4230258	5587169	8157334	12067828	15573508	28989445	32250161	53029085	52893528	61873905	75183786	124317884	150494076	178426689	253332021
8	4278760	5605261	23809612	20341704	16597911	26540923	49974589	47685782	71257666	83869469	99103570	147212160	231551122	204795478	209853587	543283351

Number of ticks a thread has waited to enter TryGetWork method

Threads \ Tile size	10	20	25	30	35	40	50	60	70	80	90	100	110	120	130	150
1	28996761	4393868	2210942	1160685	924416	554032	318116	198969	132985	114561	86084	65050	48639	48269	35188	25543
2	109338797	11826906	10980935	3295801	3138570	3419888	1952690	1784954	1475015	1004678	838166	846317	1079625	760706	742895	829705
4	377234464	24741508	15132527	13271935	12147474	8561003	6979827	5978627	6362598	4467233	4979997	4151781	3612779	4119721	3369186	3471825
6	799281171	43622460	39786969	17560152	16134436	15436618	12780984	15245762	8352171	8512016	7071339	5741215	7839001	7427045	8670213	6484582
8	1663980312	74271539	38847565	33912749	27899860	23509329	18871561	25938072	14399287	12920876	14864492	12033895	12786103	10186583	10991742	28021891

Number of times a thread has completed a computational kernel

Threads \ Tile size	10	20	25	30	35	40	50	60	70	80	90	100	110	120	130	150
1	14839408	1993082	997498	586508	404858	265184	134728	89732	56048	42938	32044	23162	16088	16088	10618	6548
2	14839408	1993082	997498	586508	404858	265184	134728	89732	56048	42938	32044	23162	16088	16088	10618	6548
4	14839408	1993082	997498	586508	404858	265184	134728	89732	56048	42938	32044	23162	16088	16088	10618	6548
6	14839408	1993082	997498	586508	404858	265184	134728	89732	56048	42938	32044	23162	16088	16088	10618	6548
8	14839408	1993082	997498	586508	404858	265184	134728	89732	56048	42938	32044	23162	16088	16088	10618	6548

Number of times a thread has received work from a secondary producer (in pipelined mode)

Threads \ Tile size	10	20	25	30	35	40	50	60	70	80	90	100	110	120	130	150
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	63	61	73	64	53	63	48	40	39	39	40	37	34	29	32	28
4	247	257	309	309	294	271	346	264	224	264	278	261	221	256	201	263
6	441	584	563	601	651	591	635	687	571	622	607	591	562	609	552	577
8	660	1022	1191	1238	1079	1155	1279	1148	1125	1050	1071	1077	1050	1127	977	994

Table 12: BTM 10x750x750 - measurement including extra statistics - collected statistics part 2

Block Tridiagonal Matrix Inverse: 10x750x750 - calculated stats - measurement including extra statistics

Speedup

Threads \ Tile size	10	20	25	30	35	40	50	60	70	80	90	100	110	120	130	150
1	0.843	0.954	0.966	0.975	0.973	0.974	0.976	0.975	0.977	0.976	0.972	0.974	0.975	0.971	0.975	0.974
2	1.103	1.694	1.806	1.880	1.893	1.887	1.926	1.935	1.938	1.937	1.934	1.933	1.936	1.929	1.927	1.918
4	1.345	2.866	3.271	3.525	3.605	3.650	3.764	3.797	3.809	3.820	3.805	3.813	3.766	3.798	3.778	3.694
6	1.412	3.694	4.421	4.982	5.147	5.258	5.504	5.567	5.586	5.612	5.615	5.618	5.538	5.483	5.462	5.355
8	1.354	4.259	5.353	6.202	6.511	6.701	7.101	7.264	7.304	7.339	7.304	7.239	7.079	7.148	7.179	6.465

Waiting time in % (#ticks waited/(#threads * #total ticks)

Threads \ Tile size	10	20	25	30	35	40	50	60	70	80	90	100	110	120	130	150
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.005%	0.009%	0.036%	0.057%	0.065%	0.081%	0.083%	0.142%	0.160%	0.262%	0.354%	0.457%	0.547%	0.514%	0.875%	1.288%
4	0.011%	0.060%	0.104%	0.138%	0.297%	0.275%	0.539%	0.726%	0.957%	1.006%	1.262%	1.419%	2.863%	1.744%	2.608%	4.813%
6	0.017%	0.102%	0.170%	0.288%	0.441%	0.584%	1.131%	1.243%	1.939%	1.890%	2.153%	2.582%	4.136%	4.778%	5.623%	7.535%
8	0.020%	0.117%	0.657%	0.670%	0.575%	0.951%	1.886%	1.799%	2.555%	2.939%	3.364%	4.886%	7.385%	6.358%	6.519%	14.631%

Spinning time in % (#ticks spinning/(#threads * #total ticks)

Threads \ Tile size	10	20	25	30	35	40	50	60	70	80	90	100	110	120	130	150
1	0.684%	0.164%	0.088%	0.048%	0.038%	0.023%	0.013%	0.008%	0.005%	0.004%	0.003%	0.002%	0.002%	0.002%	0.001%	0.001%
2	1.686%	0.393%	0.409%	0.132%	0.126%	0.138%	0.080%	0.072%	0.056%	0.037%	0.030%	0.030%	0.038%	0.025%	0.025%	0.027%
4	3.546%	0.696%	0.511%	0.497%	0.466%	0.334%	0.279%	0.236%	0.238%	0.163%	0.176%	0.145%	0.123%	0.136%	0.110%	0.107%
6	5.260%	1.054%	1.210%	0.619%	0.589%	0.578%	0.498%	0.588%	0.305%	0.304%	0.246%	0.197%	0.261%	0.236%	0.273%	0.193%
8	7.873%	1.552%	1.073%	1.117%	0.966%	0.842%	0.712%	0.978%	0.516%	0.453%	0.505%	0.399%	0.408%	0.316%	0.341%	0.755%

% times a thread asked in vain for work (#waited for work / (#waited for work + # computational kernels))

#Tiles	75	38	30	25	22	19	#VALUE!	13	11	10	9	8	7	7	6	5
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.001%	0.006%	0.010%	0.019%	0.034%	0.051%	0.088%	0.145%	0.256%	0.313%	0.481%	0.588%	0.771%	0.765%	1.347%	1.844%
4	0.002%	0.047%	0.080%	0.138%	0.405%	0.398%	0.680%	1.257%	1.641%	2.260%	2.773%	3.209%	5.001%	4.073%	6.185%	9.358%
6	0.004%	0.106%	0.220%	0.458%	0.894%	1.213%	2.011%	3.100%	4.498%	5.344%	6.534%	7.629%	10.860%	11.793%	15.151%	18.970%
8	0.006%	0.144%	0.460%	0.931%	1.570%	2.394%	3.811%	5.426%	7.731%	9.492%	11.275%	16.052%	20.753%	19.379%	23.606%	42.410%

% computational kernels pipelined = #received work from secondary producer / #total computational kernels

#Tiles	75	38	30	25	22	19	15	13	11	10	9	8	7	7	6	5
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.000%	0.003%	0.007%	0.011%	0.013%	0.024%	0.036%	0.045%	0.070%	0.091%	0.125%	0.160%	0.211%	0.180%	0.301%	0.428%
4	0.002%	0.013%	0.031%	0.053%	0.073%	0.102%	0.257%	0.294%	0.400%	0.615%	0.868%	1.127%	1.374%	1.591%	1.893%	4.016%
6	0.003%	0.029%	0.056%	0.102%	0.161%	0.223%	0.471%	0.766%	1.019%	1.449%	1.894%	2.552%	3.493%	3.785%	5.199%	8.812%
8	0.004%	0.051%	0.119%	0.211%	0.267%	0.436%	0.949%	1.279%	2.007%	2.445%	3.342%	4.650%	6.527%	7.005%	9.201%	15.180%

Table 13: BTM 10x750x750 - measurement including extra statistics - calculated statistics

Block Tridiagonal Matrix Inverse: 10x500x500

Running times

Single threaded				02:11.638												
Single threaded tiled	**01:46.023**	**01:17.099**	**01:14.395**	**01:13.197**	**01:13.643**	**01:13.691**	**01:14.714**	**01:18.254**	**01:25.019**	**01:27.179**	**01:28.443**	**01:29.663**	**01:33.710**	**01:40.589**	**01:37.581**	**01:48.711**
Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**150**
1	02:02.019	01:20.786	01:17.319	01:15.532	01:16.143	01:16.051	01:17.018	01:20.801	01:27.763	01:29.964	01:31.134	01:32.583	01:36.944	01:44.318	01:40.686	01:52.565
2	01:29.131	00:44.930	00:40.991	00:39.288	00:39.238	00:38.728	00:38.933	00:40.796	00:44.244	00:45.373	00:46.113	00:46.937	00:48.853	00:52.413	00:51.132	00:56.452
4	01:14.854	00:26.142	00:22.662	00:20.910	00:20.551	00:20.116	00:20.031	00:20.848	00:22.589	00:23.144	00:23.433	00:24.433	00:25.426	00:26.764	00:26.166	00:28.805
6	01:11.848	00:20.242	00:16.760	00:14.777	00:14.462	00:14.001	00:13.785	00:14.287	00:15.502	00:16.044	00:16.356	00:16.893	00:16.952	00:18.578	00:19.452	00:22.342
8	01:15.239	00:17.422	00:14.016	00:12.106	00:11.494	00:11.045	00:10.744	00:11.075	00:12.297	00:12.438	00:12.329	00:13.921	00:14.335	00:15.978	00:17.955	00:22.306

Number of times a thread has waited for work

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**150**
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	86	112	107	139	157	131	111	145	132	103	124	118	90	101	100	46
4	352	813	791	1174	1352	1204	847	761	714	663	553	662	613	479	426	312
6	660	2031	2355	2947	2869	2662	2356	1983	1761	1966	1688	1469	1086	1529	2156	2707
8	898	2972	4324	4738	4383	4452	4147	3502	4193	3775	3005	4659	4894	5630	6012	6922

Number of times a thread has completed a computational kernel

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**150**
1	4470458	586508	307418	192668	134728	89732	42938	32044	23162	16088	10618	6548	6548	6548	3674	3674
2	4470458	586508	307418	192668	134728	89732	42938	32044	23162	16088	10618	6548	6548	6548	3674	3674
4	4470458	586508	307418	192668	134728	89732	42938	32044	23162	16088	10618	6548	6548	6548	3674	3674
6	4470458	586508	307418	192668	134728	89732	42938	32044	23162	16088	10618	6548	6548	6548	3674	3674
8	4470458	586508	307418	192668	134728	89732	42938	32044	23162	16088	10618	6548	6548	6548	3674	3674

Number of times a thread has received work from a secondary producer (in pipelined mode)

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**150**
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	68	57	71	70	44	41	40	40	41	40	40	31	33	29	30	28
4	461	340	305	226	303	265	289	274	275	264	238	266	255	259	252	247
6	485	565	575	585	691	656	557	621	738	612	564	587	533	587	517	520
8	719	1256	1343	1161	1146	1145	1222	1092	1216	1189	859	992	983	1170	518	487

Table 14: BTM 10x500x500 - collected statistics

Block Tridiagonal Matrix Inverse: 10x500x500

Speedup

Threads \ Tile size	10	20	25	30	35	40	50	60	70	80	90	100	110	120	130	150
1	0.869	0.954	0.962	0.969	0.967	0.969	0.970	0.968	0.969	0.969	0.970	0.968	0.967	0.964	0.969	0.966
2	1.190	1.716	1.815	1.863	1.877	1.903	1.919	1.918	1.922	1.921	1.918	1.910	1.918	1.919	1.908	1.926
4	1.416	2.949	3.283	3.501	3.583	3.663	3.730	3.754	3.764	3.767	3.774	3.670	3.686	3.758	3.729	3.774
6	1.476	3.809	4.439	4.953	5.092	5.263	5.420	5.477	5.484	5.434	5.407	5.308	5.528	5.414	5.017	4.866
8	1.409	4.425	5.308	6.046	6.407	6.672	6.954	7.066	6.914	7.009	**7.174**	6.441	6.537	6.295	5.435	4.874

% times a thread asked in vain for work (#waited for work / (#waited for work + # computational kernels))

Threads\#tiles	50	25	20	17	15	13	10	9	8	7	6	5	5	5	4	4
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.002%	0.019%	0.035%	0.072%	0.116%	0.146%	0.258%	0.450%	0.567%	0.636%	1.154%	1.770%	1.356%	1.519%	2.650%	1.237%
4	0.008%	0.138%	0.257%	0.606%	0.994%	1.324%	1.934%	2.320%	2.990%	3.958%	4.950%	9.182%	8.560%	6.817%	10.390%	7.827%
6	0.015%	0.345%	0.760%	1.507%	2.085%	2.881%	5.202%	5.828%	7.066%	10.890%	13.717%	18.324%	14.226%	18.930%	36.981%	42.423%
8	0.020%	0.504%	1.387%	2.400%	3.151%	4.727%	8.807%	9.852%	15.328%	19.005%	22.058%	41.572%	42.772%	46.231%	62.069%	65.327%

% computational kernels pipelined = #received work from secondary producer / #total computational kernels

Threads\#tiles	50	25	20	17	15	13	10	9	8	7	6	5	5	5	4	4
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.002%	0.010%	0.023%	0.036%	0.033%	0.046%	0.093%	0.125%	0.177%	0.249%	0.377%	0.473%	0.504%	0.443%	0.817%	0.762%
4	0.010%	0.058%	0.099%	0.117%	0.225%	0.295%	0.673%	0.855%	1.187%	1.641%	2.241%	4.062%	3.894%	3.955%	6.859%	6.723%
6	0.011%	0.096%	0.187%	0.304%	0.513%	0.731%	1.297%	1.938%	3.186%	3.804%	5.312%	8.965%	8.140%	8.965%	14.072%	14.154%
8	0.016%	0.214%	0.437%	0.603%	0.851%	1.276%	2.846%	3.408%	5.250%	7.391%	8.090%	15.150%	15.012%	17.868%	14.099%	13.255%

Table 15: BTM 10x500x500 - calculated statistics

Block Tridiagonal Matrix Inverse: 100x100x200 - measurement including extra statistics

Running times

Single threaded non tiled **00:28.844**

Single threaded tiled	**00:35.931**	**00:29.149**	**00:29.396**	**00:29.771**	**00:30.918**	**00:31.626**	**00:34.250**	**00:44.089**

Threads\Tile size	10	20	25	30	35	40	50	80
1	00:43.680	00:31.897	00:31.840	00:31.900	00:33.021	00:33.721	00:36.410	00:46.363
2	00:35.409	00:18.323	00:17.631	00:17.022	00:17.342	00:17.571	00:18.838	00:24.085
4	00:26.509	00:11.111	00:10.029	00:09.522	00:09.524	00:09.584	00:10.479	00:17.260
6	00:25.370	00:09.206	00:07.976	00:07.392	00:07.370	00:07.616	00:08.986	00:16.276
8	00:26.789	00:08.468	00:07.175	00:06.617	00:06.876	00:07.263	00:08.793	00:16.295

Running time in ticks

Threads\Tile size	10	20	25	30	35	40	50	80
1	436796830	318965547	318398418	318998784	330213611	337207605	364104044	463627655
2	354086622	183232187	176305556	170217905	173422537	175714606	188383942	240852394
4	265093182	111112070	100293350	95222379	95238037	95835960	104789494	172596906
6	253699271	92058172	79759151	73922197	73698647	76159640	89857368	162761572
8	267890285	84684925	71747739	66170256	68758358	72627232	87929902	162954640

Number of times a thread has waited for work

Threads\Tile size	10	20	25	30	35	40	50	80
1	0	0	0	0	0	0	0	0
2	905	1177	1045	939	705	676	514	727
4	2933	5976	5630	4719	4108	4816	6108	10653
6	3889	10498	10929	12351	14887	17764	24701	24810
8	5046	12377	17459	24215	32372	37131	42357	38680

Number of ticks a thread has waited for work

Threads\Tile size	10	20	25	30	35	40	50	80
1	0	0	0	0	0	0	0	0
2	23317810	5589715	5225665	3808068	2998467	3732693	4613512	13488118
4	1716684	5741442	11679993	8607371	10753315	15155553	29291670	206661835
6	2892627	10223197	14162404	17698011	31680469	46029251	116931771	473141847
8	4329215	12533728	19059691	32792417	60858211	105644421	229119280	762393221

Number of ticks a thread has waited to enter TryGetWork method

Threads\Tile size	10	20	25	30	35	40	50	80
1	2957673	536404	294542	192498	137273	96275	62235	30312
2	19496968	5985715	5218119	5282740	4644290	4083294	3762193	2386368
4	61590818	28487372	31422423	18627873	17088280	15421740	15800857	12183751
6	142140066	52803308	57247886	49187868	39221097	39435194	30711715	22858410
8	284279706	93711314	100592991	86512226	82683584	75223380	61247381	43159139

Number of times a thread has completed a computational kernel

Threads\Tile size	10	20	25	30	35	40	50	80
1	1651403	252168	145990	92237	65228	45475	28427	11678
2	1651403	252168	145990	92237	65228	45475	28427	11678
4	1651403	252168	145990	92237	65228	45475	28427	11678
6	1651403	252168	145990	92237	65228	45475	28427	11678
8	1651403	252168	145990	92237	65228	45475	28427	11678

Number of times a thread has received work from a secondary producer (in pipelined mode)

Threads\Tile size	10	20	25	30	35	40	50	80
1	0	0	0	0	0	0	0	0
2	714	530	490	488	434	451	445	361
4	2624	2938	2683	2704	2715	2537	2156	1246
6	4321	6178	6333	6064	5642	4655	3555	1392
8	5399	11036	11606	10162	8236	6699	3965	1417

Table 16: BTM 100x100x200 - measurement including extra statistics - collected statistics.

Speedup for 100x100x200 - measurement including extra statistics

Threads\Tile size	10	20	25	30	35	40	50	80
1	0.823	0.914	0.923	0.933	0.936	0.938	0.941	0.951
2	1.015	1.591	1.667	1.749	1.783	1.800	1.818	1.831
4	1.355	2.623	2.931	3.127	3.246	3.300	3.268	2.554
6	1.416	3.166	3.686	4.027	4.195	4.153	3.811	2.709
8	1.341	3.442	4.097	4.499	4.497	4.354	3.895	2.706

Waiting time in % (#ticks waited/(#threads * #total ticks)

Threads\Tile size	10	20	25	30	35	40	50	80
1	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%
2	3.2927%	1.5253%	1.4820%	1.1186%	0.8645%	1.0621%	1.2245%	2.8001%
4	0.1619%	1.2918%	2.9115%	2.2598%	2.8227%	3.9535%	6.9882%	29.9342%
6	0.1900%	1.8509%	2.9594%	3.9902%	7.1644%	10.0730%	21.6884%	48.4494%
8	0.2020%	1.8501%	3.3206%	6.1947%	11.0638%	18.1826%	32.5713%	58.4820%

Spinning time in % (#ticks spinning/(#threads * #total ticks)

Threads\Tile size	10	20	25	30	35	40	50	80
1	0.6771%	0.1682%	0.0925%	0.0603%	0.0416%	0.0286%	0.0171%	0.0065%
2	2.7531%	1.6334%	1.4799%	1.5518%	1.3390%	1.1619%	0.9985%	0.4954%
4	5.8084%	6.4096%	7.8326%	4.8906%	4.4857%	4.0230%	3.7697%	1.7648%
6	9.3378%	9.5598%	11.9627%	11.0900%	8.8697%	8.6299%	5.6964%	2.3407%
8	13.2647%	13.8323%	17.5255%	16.3427%	15.0316%	12.9468%	8.7068%	3.3107%

% times a thread asked in vain for work (#waited for work / (#waited for work + # computational kernels))

Threads\#tiles	15	8	6	5	5	4	3	2
1	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%
2	0.0548%	0.4646%	0.7107%	1.0078%	1.0693%	1.4648%	1.7760%	5.8605%
4	0.1773%	2.3150%	3.7132%	4.8672%	5.9248%	9.5763%	17.6864%	47.7050%
6	0.2349%	3.9967%	6.9647%	11.8092%	18.5820%	28.0903%	46.4934%	67.9950%
8	0.3046%	4.6786%	10.6816%	20.7940%	33.1680%	44.9495%	59.8398%	76.8100%

% computational kernels pipelined = #received work from secondary producer / #total computational kernels

Threads\#tiles	15	8	6	5	5	4	3	2
1	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%	0.0000%
2	0.0432%	0.2102%	0.3356%	0.5291%	0.6654%	0.9918%	1.5654%	3.0913%
4	0.1589%	1.1651%	1.8378%	2.9316%	4.1623%	5.5789%	7.5843%	10.6696%
6	0.2617%	2.4500%	4.3380%	6.5744%	8.6497%	10.2364%	12.5057%	11.9198%
8	0.3269%	4.3764%	7.9499%	11.0173%	12.6265%	14.7312%	13.9480%	12.1339%

Table 17: BTM 100x100x200 - measurement including extra statistics - calculated statistics

Block Tridiagonal Matrix Inverse: 100x100x200

Running times

Single threaded non tiled 00:28.844

Single threaded tiled	**00:35.931**	**00:29.149**	**00:29.396**	**00:29.771**	**00:30.918**	**00:31.626**	**00:34.250**	**00:44.089**
Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	00:43.429	00:31.784	00:31.788	00:31.836	00:33.008	00:33.769	00:36.375	00:46.351
2	00:33.037	00:18.446	00:17.729	00:16.969	00:17.398	00:17.635	00:18.805	00:24.173
4	00:26.486	00:11.161	00:10.124	00:09.527	00:09.531	00:09.607	00:10.477	00:17.273
6	00:25.338	00:09.133	00:08.072	00:07.384	00:07.387	00:07.608	00:08.978	00:16.252
8	00:26.868	00:08.435	00:07.230	00:06.597	00:06.802	00:07.527	00:08.809	00:16.364

Number of times a thread has waited for work

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	0	0	0	0	0	0	0	0
2	927	1206	1052	927	697	680	517	723
4	3134	6163	5601	4702	4148	4830	6192	10753
6	4251	10689	11193	12513	15105	18311	24870	24932
8	5471	13204	17830	24653	32676	38343	42820	38784

Number of times a thread has completed a computational kernel

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	1651403	252168	145990	92237	65228	45475	28427	11678
2	1651403	252168	145990	92237	65228	45475	28427	11678
4	1651403	252168	145990	92237	65228	45475	28427	11678
6	1651403	252168	145990	92237	65228	45475	28427	11678
8	1651403	252168	145990	92237	65228	45475	28427	11678

Number of times a thread has received work from a secondary producer (in pipelined mode)

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	0	0	0	0	0	0	0	0
2	704	515	491	492	449	439	432	363
4	2588	2941	2697	2692	2699	2551	2181	1239
6	4771	6276	6374	6102	5629	4689	3593	1379
8	7576	11302	11803	10078	8385	6577	3888	1414

% times a thread asked in vain for work (#waited for work / (#waited for work + # computational kernels))

Threads \ #tiles	**15**	**8**	**6**	**5**	**5**	**4**	**3**	**2**
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.056%	0.476%	0.715%	0.995%	1.057%	1.473%	1.786%	5.830%
4	0.189%	2.386%	3.695%	4.850%	5.979%	9.601%	17.886%	47.938%
6	0.257%	4.066%	7.121%	11.946%	18.803%	28.707%	46.663%	68.102%
8	0.330%	4.976%	10.884%	21.091%	33.376%	45.746%	60.101%	76.858%

% computational kernels pipelined = #received work from secondary producer / #total computational kernels

Threads \ #tiles	**15**	**8**	**6**	**5**	**5**	**4**	**3**	**2**
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.043%	0.204%	0.336%	0.533%	0.688%	0.965%	1.520%	3.108%
4	0.157%	1.166%	1.847%	2.919%	4.138%	5.610%	7.672%	10.610%
6	0.289%	2.489%	4.366%	6.616%	8.630%	10.311%	12.639%	11.809%
8	0.459%	4.482%	8.085%	10.926%	12.855%	14.463%	13.677%	12.108%

Speedup

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	0.827	0.917	0.925	0.935	0.937	0.937	0.942	0.951
2	1.088	1.580	1.658	1.754	1.777	1.793	1.821	1.824
4	1.357	2.612	2.904	3.125	3.244	3.292	3.269	2.552
6	1.418	3.192	3.642	4.032	4.185	4.157	3.815	2.713
8	1.337	3.456	4.066	4.513	**4.545**	4.202	3.888	2.694

Table 18: BTM 100x100x200 - collected and calculated statistics

Block Tridiagonal Matrix Inverse: 100x50x100

Running times

Single threaded 00:03.580

Single threaded tiled	**00:05.130**	**00:04.654**	**00:04.836**	**00:05.089**	**00:05.262**	**00:05.750**	**00:06.248**	**00:08.925**
Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	00:06.655	00:05.480	00:05.711	00:05.810	00:05.943	00:06.434	00:06.893	00:09.493
2	00:05.726	00:03.486	00:03.353	00:03.391	00:03.450	00:03.804	00:04.350	00:07.449
4	00:03.934	00:02.596	00:02.388	00:02.506	00:02.680	00:02.857	00:03.404	00:06.165
6	00:03.722	00:02.368	00:02.383	00:02.537	00:02.672	00:02.966	00:03.673	00:06.363
8	00:04.204	00:02.688	00:02.665	00:02.754	00:03.095	00:03.276	00:03.956	00:06.659

Number of times a thread has waited for work

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	0	0	0	0	0	0	0	0
2	980	654	469	476	487	589	907	1393
4	3114	3659	4600	6975	8490	9177	9539	5748
6	3825	9261	14913	17053	18500	19494	17493	10014
8	4166	16100	21558	23735	26030	27163	24250	13088

Number of times a thread has completed a computational kernel

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	249083	46203	28834	19178	13634	11622	7383	3838
2	249083	46203	28834	19178	13634	11622	7383	3838
4	249083	46203	28834	19178	13634	11622	7383	3838
6	249083	46203	28834	19178	13634	11622	7383	3838
8	249083	46203	28834	19178	13634	11622	7383	3838

Number of times a thread has received work from a secondary producer (in pipelined mode)

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	0	0	0	0	0	0	0	0
2	713	527	567	506	426	405	398	132
4	2694	2573	2205	1850	1380	1278	706	288
6	4822	5390	3818	2481	1676	1430	748	383
8	6405	7551	4019	2660	1741	1444	778	454

% times a thread asked in vain for work (#waited for work / (#waited for work + # computational kernels))

Threads \ #tiles	**8**	**4**	**3**	**3**	**3**	**2**	**2**	**1**
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.392%	1.396%	1.601%	2.422%	3.449%	4.824%	10.941%	26.630%
4	1.235%	7.338%	13.758%	26.670%	38.375%	44.122%	56.370%	59.962%
6	1.512%	16.697%	34.089%	47.067%	57.571%	62.649%	70.321%	72.293%
8	1.645%	25.841%	42.781%	55.310%	65.626%	70.035%	76.660%	77.325%

% computational kernels pipelined = #received work from secondary producer / #total computational kernels

Threads \ #tiles	**8**	**4**	**3**	**3**	**3**	**2**	**2**	**1**
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.286%	1.141%	1.966%	2.638%	3.125%	3.485%	5.391%	3.439%
4	1.082%	5.569%	7.647%	9.646%	10.122%	10.996%	9.563%	7.504%
6	1.936%	11.666%	13.241%	12.937%	12.293%	12.304%	10.131%	9.979%
8	2.571%	16.343%	13.938%	13.870%	12.770%	12.425%	10.538%	11.829%

Speedup

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	0.771	0.849	0.847	0.876	0.885	0.894	0.906	0.940
2	0.896	1.335	1.442	1.501	1.525	1.512	1.436	1.198
4	1.304	1.793	2.025	**2.031**	1.963	2.013	1.835	1.448
6	1.378	1.965	2.029	2.006	1.969	1.939	1.701	1.403
8	1.220	1.731	1.815	1.848	1.700	1.755	1.579	1.340

Table 19: BTM 100x50x100 - collected and calculated statistics

Block Tridiagonal Matrix Inverse: 100x150x250

Running times

Single threaded			01:05.174					
Single threaded tiled	**01:20.381**	**01:03.626**	**01:02.792**	**01:02.916**	**01:04.203**	**01:05.476**	**01:09.813**	**01:26.184**
Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	01:34.607	01:07.534	01:06.342	01:06.060	01:07.404	01:09.196	01:13.347	01:30.349
2	01:10.393	00:38.385	00:36.193	00:35.096	00:35.206	00:35.757	00:37.670	00:46.128
4	00:58.738	00:22.911	00:20.393	00:19.132	00:18.932	00:19.145	00:19.823	00:28.090
6	00:55.554	00:18.505	00:15.785	00:14.288	00:13.990	00:14.023	00:14.711	00:25.014
8	00:58.329	00:16.875	00:13.674	00:12.059	00:11.728	00:11.753	00:13.309	00:24.353

Number of times a thread has waited for work

Threads \ Tile size	10	20	25	30	35	40	50	80
1	0	0	0	0	0	0	0	0
2	951	1358	1348	1180	1074	941	693	545
4	3174	8174	7435	6373	5868	5168	4156	9780
6	4960	13554	14569	14382	13835	13736	17908	28221
8	5920	15534	20370	23481	26835	31794	45561	46373

Number of times a thread has completed a computational kernel

Threads \ Tile size	10	20	25	30	35	40	50	80
1	3570840	531437	296253	186817	124708	89590	53670	18683
2	3570840	531437	296253	186817	124708	89590	53670	18683
4	3570840	531437	296253	186817	124708	89590	53670	18683
6	3570840	531437	296253	186817	124708	89590	53670	18683
8	3570840	531437	296253	186817	124708	89590	53670	18683

Number of times a thread has received work from a secondary producer (in pipelined

Threads \ Tile size	10	20	25	30	35	40	50	80
1	0	0	0	0	0	0	0	0
2	730	535	478	469	432	440	421	416
4	2485	2953	2829	2784	2738	2716	2639	1790
6	4651	6328	6471	6385	6207	6099	5385	2381
8	6224	11377	11796	11900	11381	10405	8078	2645

% times a thread asked in vain for work (#waited for work / (#waited for work + # computational kernels))

Threads \ #tiles	20	10	8	7	6	5	4	3
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.027%	0.255%	0.453%	0.628%	0.854%	1.039%	1.275%	2.834%
4	0.089%	1.515%	2.448%	3.299%	4.494%	5.454%	7.187%	34.360%
6	0.139%	2.487%	4.687%	7.148%	9.986%	13.294%	25.019%	60.168%
8	0.166%	2.840%	6.434%	11.166%	17.708%	26.193%	45.914%	71.282%

% computational kernels pipelined = #received work from secondary producer / #total computational kernels

Threads \ #tiles	20	10	8	7	6	5	4	3
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.020%	0.101%	0.161%	0.251%	0.346%	0.491%	0.784%	2.227%
4	0.070%	0.556%	0.955%	1.490%	2.196%	3.032%	4.917%	9.581%
6	0.130%	1.191%	2.184%	3.418%	4.977%	6.808%	10.034%	12.744%
8	0.174%	2.141%	3.982%	6.370%	9.126%	11.614%	15.051%	14.157%

Speedup

Threads \ Tile size	10	20	25	30	35	40	50	80
1	0.850	0.942	0.946	0.952	0.953	0.946	0.952	0.954
2	1.142	1.658	1.735	1.793	1.824	1.831	1.853	1.868
4	1.368	2.777	3.079	3.289	3.391	3.420	3.522	3.068
6	1.447	3.438	3.978	4.403	4.589	4.669	4.746	3.445
8	1.378	3.770	4.592	5.217	5.474	**5.571**	5.246	3.539

Table 20: BTM 100x150x250 - collected and calculated statistics

Block Tridiagonal Matrix Inverse: 200x100x200

Running times

	Single threaded		01:01.022					
Single threaded tiled	**01:17.464**	**01:03.200**	**01:02.677**	**01:03.449**	**01:05.466**	**01:06.956**	**01:11.692**	**01:33.501**
Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	01:32.749	01:07.685	01:07.170	01:07.336	01:09.692	01:11.420	01:16.030	01:38.308
2	01:13.440	00:38.841	00:37.338	00:36.016	00:36.918	00:37.233	00:39.304	00:51.107
4	00:57.117	00:24.226	00:21.564	00:20.204	00:20.276	00:20.414	00:21.848	00:36.057
6	00:54.501	00:19.784	00:17.366	00:15.693	00:15.578	00:16.115	00:18.588	00:33.771
8	00:57.565	00:18.286	00:15.523	00:14.096	00:14.220	00:15.195	00:18.224	00:33.936

Number of times a thread has waited for work

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	0	0	0	0	0	0	0	0
2	1877	2461	2184	1922	1533	1380	1121	1390
4	6436	12934	11367	9896	8684	9540	11906	21617
6	8568	21403	23030	25510	29067	35532	48849	51184
8	10422	25705	36579	49136	64410	77364	85808	80608

Number of times a thread has completed a computational kernel

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	3502566	543984	303448	196180	137217	96742	57702	24944
2	3502566	543984	303448	196180	137217	96742	57702	24944
4	3502566	543984	303448	196180	137217	96742	57702	24944
6	3502566	543984	303448	196180	137217	96742	57702	24944
8	3502566	543984	303448	196180	137217	96742	57702	24944

Number of times a thread has received work from a secondary producer (in pipelined

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	0	0	0	0	0	0	0	0
2	1470	1049	962	921	902	911	868	747
4	5193	5771	5338	5579	5408	5047	4271	2767
6	9555	12476	12626	12481	11382	9891	7155	3050
8	13157	22637	23719	21099	17272	14146	7893	3101

% times a thread asked in vain for work (#waited for work / (#waited for work + # computational kernels))

Threads \ #tiles	**15**	**8**	**6**	**5**	**5**	**4**	**3**	**2**
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.054%	0.450%	0.715%	0.970%	1.105%	1.406%	1.906%	5.278%
4	0.183%	2.322%	3.611%	4.802%	5.952%	8.976%	17.104%	46.427%
6	0.244%	3.786%	7.054%	11.507%	17.480%	26.862%	45.846%	67.234%
8	0.297%	4.512%	10.758%	20.030%	31.945%	44.435%	59.792%	76.368%

% computational kernels pipelined = #received work from secondary producer / #total computational kernels

Threads \ #tiles	**15**	**8**	**6**	**5**	**5**	**4**	**3**	**2**
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.042%	0.193%	0.317%	0.469%	0.657%	0.942%	1.504%	2.995%
4	0.148%	1.061%	1.759%	2.844%	3.941%	5.217%	7.402%	11.093%
6	0.273%	2.293%	4.161%	6.362%	8.295%	10.224%	12.400%	12.227%
8	0.376%	4.161%	7.816%	10.755%	12.587%	14.622%	13.679%	12.432%

Speedup

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	0.835	0.934	0.933	0.942	0.939	0.937	0.943	0.951
2	1.055	1.627	1.679	1.762	1.773	1.798	1.824	1.830
4	1.356	2.609	2.907	3.140	3.229	3.280	3.281	2.593
6	1.421	3.195	3.609	4.043	4.202	4.155	3.857	2.769
8	1.346	3.456	4.038	4.501	**4.604**	4.406	3.934	2.755

Table 21: BTM 200x100x200 - collected and calculated statistics

Block Tridiagonal Matrix Inverse: 50x100x200

Running times

Single threaded			00:15.105					
Single threaded tiled	**00:18.724**	**00:15.308**	**00:15.325**	**00:15.569**	**00:16.029**	**00:16.541**	**00:17.885**	**00:23.361**
Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	00:22.315	00:17.124	00:16.970	00:17.567	00:17.386	00:17.822	00:19.075	00:24.551
2	00:15.313	00:10.257	00:09.391	00:11.141	00:09.420	00:09.316	00:10.018	00:12.770
4	00:11.751	00:06.374	00:05.784	00:06.950	00:05.264	00:05.274	00:05.534	00:09.079
6	00:11.067	00:05.189	00:04.754	00:05.692	00:04.160	00:04.277	00:04.725	00:08.510
8	00:11.690	00:04.896	00:04.334	00:05.535	00:03.850	00:03.885	00:04.636	00:08.649

Number of times a thread has waited for work

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	0	0	0	0	0	0	0	0
2	473	624	557	467	364	318	247	341
4	1566	3246	2820	2364	2232	2375	2809	5284
6	2087	5501	5763	6317	7019	9079	12366	12794
8	2592	6827	8989	11955	16262	19139	22146	19390

Number of times a thread has completed a computational kernel

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	868587	136561	75543	49493	33507	24185	14933	6194
2	868587	136561	75543	49493	33507	24185	14933	6194
4	868587	136561	75543	49493	33507	24185	14933	6194
6	868587	136561	75543	49493	33507	24185	14933	6194
8	868587	136561	75543	49493	33507	24185	14933	6194

Number of times a thread has received work from a secondary producer (in pipelined

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	0	0	0	0	0	0	0	0
2	366	257	230	239	239	252	222	197
4	1261	1438	1350	1341	1340	1246	1120	694
6	2286	3228	3148	3101	2802	2513	1851	794
8	2903	5794	5979	5216	4308	3676	2000	821

% times a thread asked in vain for work (#waited for work / (#waited for work + # computational kernels))

Threads \ #tiles	**15**	**8**	**6**	**5**	**5**	**4**	**3**	**2**
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.054%	0.455%	0.732%	0.935%	1.075%	1.298%	1.627%	5.218%
4	0.180%	2.322%	3.599%	4.559%	6.245%	8.942%	15.832%	46.036%
6	0.240%	3.872%	7.088%	11.319%	17.320%	27.294%	45.298%	67.379%
8	0.298%	4.761%	10.634%	19.455%	32.675%	44.176%	59.727%	75.790%

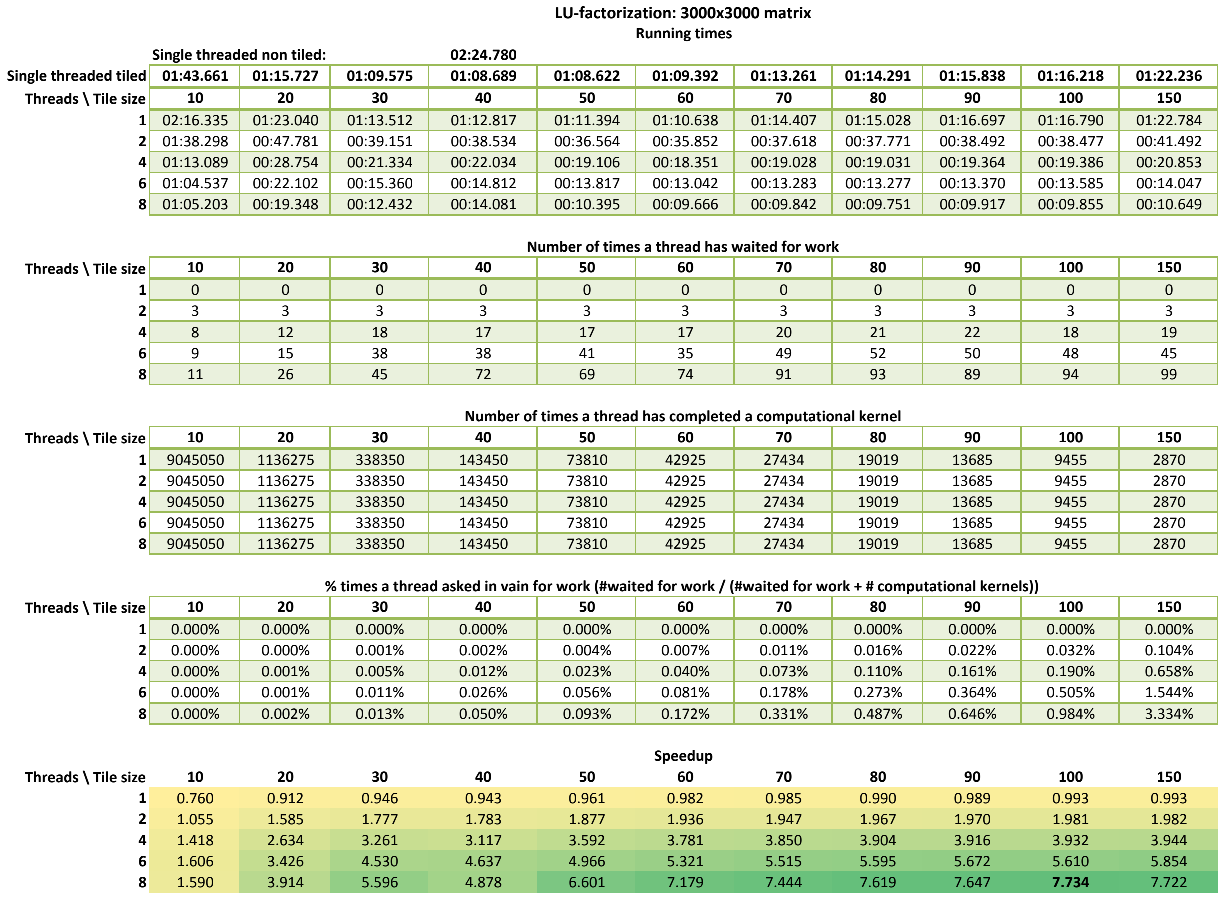
% computational kernels pipelined = #received work from secondary producer / #total computational kernels

Threads \ #tiles	**15**	**8**	**6**	**5**	**5**	**4**	**3**	**2**
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.042%	0.188%	0.304%	0.483%	0.713%	1.042%	1.487%	3.180%
4	0.145%	1.053%	1.787%	2.709%	3.999%	5.152%	7.500%	11.204%
6	0.263%	2.364%	4.167%	6.266%	8.362%	10.391%	12.395%	12.819%
8	0.334%	4.243%	7.915%	10.539%	12.857%	15.200%	13.393%	13.255%

Speedup

Threads \ Tile size	**10**	**20**	**25**	**30**	**35**	**40**	**50**	**80**
1	0.839	0.894	0.903	0.886	0.922	0.928	0.938	0.952
2	1.223	1.492	1.632	1.397	1.702	1.776	1.785	1.829
4	1.593	2.402	2.650	2.240	3.045	3.136	3.232	2.573
6	1.692	2.950	3.224	2.735	3.853	3.867	3.785	2.745
8	1.602	3.127	3.536	2.813	4.163	**4.258**	3.858	2.701

Table 22: BTM 50x100x200 - collected and calculated statistics



LU-factorization: 3000x3000 matrix

Running times

	Single threaded non tiled:			**02:24.780**							
Single threaded tiled	**01:43.661**	**01:15.727**	**01:09.575**	**01:08.689**	**01:08.622**	**01:09.392**	**01:13.261**	**01:14.291**	**01:15.838**	**01:16.218**	**01:22.236**
Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	02:16.335	01:23.040	01:13.512	01:12.817	01:11.394	01:10.638	01:14.407	01:15.028	01:16.697	01:16.790	01:22.784
2	01:38.298	00:47.781	00:39.151	00:38.534	00:36.564	00:35.852	00:37.618	00:37.771	00:38.492	00:38.477	00:41.492
4	01:13.089	00:28.754	00:21.334	00:22.034	00:19.106	00:18.351	00:19.028	00:19.031	00:19.364	00:19.386	00:20.853
6	01:04.537	00:22.102	00:15.360	00:14.812	00:13.817	00:13.042	00:13.283	00:13.277	00:13.370	00:13.585	00:14.047
8	01:05.203	00:19.348	00:12.432	00:14.081	00:10.395	00:09.666	00:09.842	00:09.751	00:09.917	00:09.855	00:10.649

Number of times a thread has waited for work

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	0	0	0	0	0	0	0	0	0	0	0
2	3	3	3	3	3	3	3	3	3	3	3
4	8	12	18	17	17	17	20	21	22	18	19
6	9	15	38	38	41	35	49	52	50	48	45
8	11	26	45	72	69	74	91	93	89	94	99

Number of times a thread has completed a computational kernel

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	9045050	1136275	338350	143450	73810	42925	27434	19019	13685	9455	2870
2	9045050	1136275	338350	143450	73810	42925	27434	19019	13685	9455	2870
4	9045050	1136275	338350	143450	73810	42925	27434	19019	13685	9455	2870
6	9045050	1136275	338350	143450	73810	42925	27434	19019	13685	9455	2870
8	9045050	1136275	338350	143450	73810	42925	27434	19019	13685	9455	2870

% times a thread asked in vain for work (#waited for work / (#waited for work + # computational kernels))

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.000%	0.000%	0.001%	0.002%	0.004%	0.007%	0.011%	0.016%	0.022%	0.032%	0.104%
4	0.000%	0.001%	0.005%	0.012%	0.023%	0.040%	0.073%	0.110%	0.161%	0.190%	0.658%
6	0.000%	0.001%	0.011%	0.026%	0.056%	0.081%	0.178%	0.273%	0.364%	0.505%	1.544%
8	0.000%	0.002%	0.013%	0.050%	0.093%	0.172%	0.331%	0.487%	0.646%	0.984%	3.334%

Speedup

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	0.760	0.912	0.946	0.943	0.961	0.982	0.985	0.990	0.989	0.993	0.993
2	1.055	1.585	1.777	1.783	1.877	1.936	1.947	1.967	1.970	1.981	1.982
4	1.418	2.634	3.261	3.117	3.592	3.781	3.850	3.904	3.916	3.932	3.944
6	1.606	3.426	4.530	4.637	4.966	5.321	5.515	5.595	5.672	5.610	5.854
8	1.590	3.914	5.596	4.878	6.601	7.179	7.444	7.619	7.647	**7.734**	7.722

Table 23: LU-factorization: 3000x3000 - collected and calculated statistics

Inverse: 2500x2500 matrix

Running times

Single threaded non tiled:				**08:38.534**							
Single threaded tiled	**03:07.846**	**02:14.102**	**02:02.850**	**01:59.780**	**01:59.983**	**02:00.532**	**02:07.749**	**02:08.967**	**02:11.261**	**02:12.450**	**02:23.733**
Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	02:29.847	01:34.949	01:24.777	01:21.720	01:21.399	01:21.296	01:25.870	01:26.448	01:28.097	01:28.810	01:36.365
2	01:49.892	00:54.776	00:44.943	00:44.251	00:41.698	00:41.204	00:43.116	00:43.389	00:44.262	00:44.555	00:48.307
4	01:20.419	00:31.873	00:24.354	00:22.364	00:21.271	00:21.064	00:21.833	00:21.911	00:22.282	00:22.423	00:24.492
6	01:10.308	00:24.489	00:17.429	00:16.707	00:14.620	00:14.316	00:14.804	00:14.829	00:15.025	00:15.135	00:16.676
8	01:12.190	00:21.046	00:13.827	00:12.595	00:11.340	00:10.956	00:11.290	00:11.217	00:11.379	00:11.640	00:12.796

Number of times a thread has waited for work

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	0	0	0	0	0	0	0	0	0	0	0
2	3	3	4	4	3	4	4	5	4	3	4
4	10	26	54	40	27	42	35	126	31	28	50
6	20	61	208	222	88	185	161	328	192	90	160
8	23	74	300	319	190	459	506	289	173	366	321

Number of times a thread has completed a computational kernel

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	10479250	1317750	402220	170688	85850	51170	32412	22880	15428	11050	3570
2	10479250	1317750	402220	170688	85850	51170	32412	22880	15428	11050	3570
4	10479250	1317750	402220	170688	85850	51170	32412	22880	15428	11050	3570
6	10479250	1317750	402220	170688	85850	51170	32412	22880	15428	11050	3570
8	10479250	1317750	402220	170688	85850	51170	32412	22880	15428	11050	3570

% times a thread asked in vain for work (#waited for work / (#waited for work + # computational kernels))

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.000%	0.000%	0.001%	0.002%	0.003%	0.008%	0.012%	0.022%	0.026%	0.027%	0.112%
4	0.000%	0.002%	0.013%	0.023%	0.031%	0.082%	0.108%	0.548%	0.201%	0.253%	1.381%
6	0.000%	0.005%	0.052%	0.130%	0.102%	0.360%	0.494%	1.413%	1.229%	0.808%	4.290%
8	0.000%	0.006%	0.075%	0.187%	0.221%	0.889%	1.537%	1.247%	1.109%	3.206%	8.250%

Speedup

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	1.254	1.412	1.449	1.466	1.474	1.483	1.488	1.492	1.490	1.491	1.492
2	1.709	2.448	2.733	2.707	2.877	2.925	2.963	2.972	2.966	2.973	2.975
4	2.336	4.207	5.044	5.356	5.641	5.722	5.851	5.886	5.891	5.907	5.869
6	2.672	5.476	7.049	7.169	8.207	8.419	8.629	8.697	8.736	8.751	8.619
8	2.602	6.372	8.885	9.510	10.581	11.001	11.315	11.497	11.535	11.379	11.233

Speedup relative to 1 thread

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	1.364	1.733	1.886	1.847	1.952	1.973	1.992	1.992	1.990	1.993	1.995
4	1.863	2.979	3.481	3.654	3.827	3.859	3.933	3.945	3.954	3.961	3.935
6	2.131	3.877	4.864	4.891	5.568	5.679	5.800	5.830	5.863	5.868	5.779
8	2.076	4.511	6.131	6.488	7.178	7.420	7.606	7.707	7.742	7.630	7.531

Table 24: Inverse: 2500x2500 - collected and calculated statistics

Minus -- Matrix -- Inverse-Matrix Multiply: 2500x2500 matrix

Running times

Single threaded non tiled: **13:35.111**

Single threaded tiled	**04:31.700**	**03:06.800**	**02:46.100**	**02:41.700**	**02:40.300**	**02:40.900**	**02:48.400**	**02:50.200**	**02:52.000**	**02:53.000**	**03:04.600**
Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	03:48.649	02:19.881	02:05.970	02:02.404	02:00.583	02:00.544	02:06.090	02:07.014	02:08.584	02:09.520	02:17.737
2	02:39.952	01:21.536	01:06.562	01:02.993	01:00.773	01:00.642	01:03.222	01:03.723	01:04.440	01:04.840	01:09.074
4	02:01.768	00:48.250	00:36.084	00:36.453	00:31.903	00:31.149	00:32.054	00:32.120	00:32.417	00:32.592	00:34.880
6	01:47.283	00:37.039	00:25.863	00:25.837	00:21.589	00:21.249	00:21.651	00:21.665	00:21.841	00:21.935	00:23.423
8	01:49.384	00:31.868	00:20.914	00:21.623	00:17.064	00:16.295	00:16.500	00:16.457	00:16.545	00:16.748	00:17.965

Number of times a thread has waited for work

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	0	0	0	0	0	0	0	0	0	0	0
2	3	4	4	4	4	4	4	5	4	3	4
4	9	24	55	37	25	39	41	140	36	24	49
6	18	57	180	172	96	187	198	344	213	108	96
8	27	100	375	425	203	526	537	611	180	443	291

Number of times a thread has completed a computational kernel

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	15687500	1968750	599760	254016	127500	75852	47952	33792	22736	16250	5202
2	15687500	1968750	599760	254016	127500	75852	47952	33792	22736	16250	5202
4	15687500	1968750	599760	254016	127500	75852	47952	33792	22736	16250	5202
6	15687500	1968750	599760	254016	127500	75852	47952	33792	22736	16250	5202
8	15687500	1968750	599760	254016	127500	75852	47952	33792	22736	16250	5202

% times a thread asked in vain for work (#waited for work / (#waited for work + # computational kernels))

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.000%	0.000%	0.001%	0.002%	0.003%	0.005%	0.008%	0.015%	0.018%	0.018%	0.077%
4	0.000%	0.001%	0.009%	0.015%	0.020%	0.051%	0.085%	0.413%	0.158%	0.147%	0.933%
6	0.000%	0.003%	0.030%	0.068%	0.075%	0.246%	0.411%	1.008%	0.928%	0.660%	1.812%
8	0.000%	0.005%	0.062%	0.167%	0.159%	0.689%	1.107%	1.776%	0.785%	2.654%	5.298%

Speedup

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	1.188	1.335	1.319	1.321	1.329	1.335	1.336	1.340	1.338	1.336	1.340
2	1.699	2.291	2.495	2.567	2.638	2.653	2.664	2.671	2.669	2.668	2.672
4	2.231	3.872	4.603	4.436	5.025	5.165	5.254	5.299	5.306	5.308	5.292
6	2.533	5.043	6.422	6.258	7.425	7.572	7.778	7.856	7.875	7.887	7.881
8	2.484	5.862	7.942	7.478	9.394	9.874	10.206	10.342	10.396	10.330	10.276

Speedup relative to 1 thread

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	1.429	1.716	1.893	1.943	1.984	1.988	1.994	1.993	1.995	1.998	1.994
4	1.878	2.899	3.491	3.358	3.780	3.870	3.934	3.954	3.967	3.974	3.949
6	2.131	3.777	4.871	4.738	5.585	5.673	5.824	5.863	5.887	5.905	5.880
8	2.090	4.389	6.023	5.661	7.067	7.398	7.642	7.718	7.772	7.733	7.667

Table 25: Minus – Matrix – Inverse-Matrix Multiply: 2500x2500 - collected and calculated statistics

MinusPlusPlus: 5000x5000 matrix

Running times

Single threaded non tiled: 00:03.925

Single threaded tiled	**00:05.500**	**00:01.900**	**00:01.100**	**00:01.500**	**00:00.900**	**00:01.000**	**00:00.800**	**00:00.800**	**00:00.800**	**00:00.800**	**00:00.800**
	10	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	00:02.802	00:01.673	00:01.336	00:01.226	00:01.078	00:01.085	00:00.819	00:00.728	00:00.729	00:00.705	00:00.768
2	00:02.419	00:01.489	00:01.260	00:00.971	00:00.548	00:00.506	00:00.457	00:00.572	00:00.702	00:00.545	00:00.592
4	00:02.282	00:00.814	00:00.501	00:01.052	00:01.005	00:00.780	00:00.678	00:00.403	00:00.400	00:00.377	00:00.383
6	00:02.025	00:00.885	00:00.575	00:00.589	00:00.496	00:01.574	00:01.248	00:00.714	00:00.815	00:02.026	00:03.321
8	00:14.704	00:08.145	00:02.890	00:18.513	00:07.635	00:06.193	00:01.216	00:04.165	00:00.942	00:00.639	00:07.579

Number of times a thread has waited for work

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0

Number of times a thread has completed a computational kernel

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	250000	62500	27889	15625	10000	7056	5184	3969	3136	2500	1156
2	250000	62500	27889	15625	10000	7056	5184	3969	3136	2500	1156
4	250000	62500	27889	15625	10000	7056	5184	3969	3136	2500	1156
6	250000	62500	27889	15625	10000	7056	5184	3969	3136	2500	1156
8	250000	62500	27889	15625	10000	7056	5184	3969	3136	2500	1156

% times a thread asked in vain for work (#waited for work / (#waited for work + # computational kernels))

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
4	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
6	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
8	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%

Speedup

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	1.963	1.136	0.823	1.223	0.835	0.922	0.977	1.099	1.097	1.135	1.042
2	2.274	1.276	0.873	1.545	1.642	1.976	1.751	1.399	1.140	1.468	1.351
4	2.410	2.334	2.196	1.426	0.896	1.282	1.180	1.985	2.000	2.122	2.089
6	**2.716**	2.147	1.913	2.547	1.815	0.635	0.641	1.120	0.982	0.395	0.241
8	0.374	0.233	0.381	0.081	0.118	0.161	0.658	0.192	0.849	1.252	0.106

Table 26: MinusPlusPlus: 5000x5000 - collected and calculated statistics

Multiply: 2500x2500 matrix

Running times

Single threaded non tiled: 04:56.007

Single threaded tiled	**02:52.900**	**02:06.000**	**01:54.200**	**01:49.500**	**01:47.500**	**01:46.000**	**01:50.200**	**01:48.700**	**01:49.100**	**01:47.800**	**01:46.800**
Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	02:55.451	02:05.434	01:55.437	01:50.281	01:48.125	01:46.246	01:50.393	01:48.842	01:49.245	01:47.783	01:46.885
2	02:01.216	01:12.963	01:00.892	00:57.005	00:55.397	00:53.982	00:55.705	00:54.699	00:54.825	00:54.149	00:53.550
4	01:29.969	00:43.108	00:32.978	00:30.154	00:28.413	00:27.510	00:28.139	00:27.570	00:27.523	00:27.274	00:26.847
6	01:31.483	00:32.876	00:23.668	00:20.436	00:19.213	00:18.771	00:18.924	00:18.552	00:18.504	00:18.302	00:18.011
8	01:39.140	00:28.106	00:19.186	00:15.965	00:14.813	00:14.259	00:14.413	00:14.095	00:13.941	00:13.898	00:13.591

Number of times a thread has waited for work

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0

Number of times a thread has completed a computational kernel

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	62500	15625	7056	3969	2500	1764	1296	1024	784	625	289
2	62500	15625	7056	3969	2500	1764	1296	1024	784	625	289
4	62500	15625	7056	3969	2500	1764	1296	1024	784	625	289
6	62500	15625	7056	3969	2500	1764	1296	1024	784	625	289
8	62500	15625	7056	3969	2500	1764	1296	1024	784	625	289

% times a thread asked in vain for work (#waited for work / (#waited for work + # computational kernels))

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
4	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
6	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
8	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%

Speedup

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	0.985	1.005	0.989	0.993	0.994	0.998	0.998	0.999	0.999	1.000	0.999
2	1.426	1.727	1.875	1.921	1.941	1.964	1.978	1.987	1.990	1.991	1.994
4	1.922	2.923	3.463	3.631	3.783	3.853	3.916	3.943	3.964	3.952	3.978
6	1.890	3.833	4.825	5.358	5.595	5.647	5.823	5.859	5.896	5.890	5.930
8	1.744	4.483	5.952	6.859	7.257	7.434	7.646	7.712	7.826	7.757	**7.858**

Table 27: Multiply: 2500x2500 - collected and calculated statistics

PlusMultiply: 2500x2500 matrix

Running times

Single threaded non tiled: 04:53.947

Single threaded tiled	**02:56.276**	**02:06.550**	**01:54.113**	**01:49.166**	**01:47.437**	**01:45.900**	**01:50.199**	**01:48.668**	**01:49.113**	**01:47.722**	**01:46.783**
Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	02:56.258	02:05.783	01:55.586	01:50.872	01:48.585	01:46.801	01:50.434	01:48.892	01:49.085	01:47.794	01:46.643
2	02:04.356	01:12.673	01:00.871	00:56.793	00:55.448	00:54.213	00:55.721	00:54.749	00:54.885	00:54.249	00:53.628
4	01:31.534	00:43.715	00:33.269	00:31.121	00:28.851	00:27.786	00:28.211	00:27.696	00:27.615	00:27.297	00:26.898
6	01:33.945	00:33.725	00:23.748	00:23.333	00:19.766	00:18.895	00:19.068	00:18.579	00:18.607	00:18.354	00:18.053
8	01:41.400	00:28.703	00:19.306	00:17.179	00:15.314	00:14.527	00:14.507	00:14.085	00:14.029	00:13.969	00:13.624

Number of times a thread has waited for work

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0

Number of times a thread has completed a computational kernel

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	62500	15625	7056	3969	2500	1764	1296	1024	784	625	289
2	62500	15625	7056	3969	2500	1764	1296	1024	784	625	289
4	62500	15625	7056	3969	2500	1764	1296	1024	784	625	289
6	62500	15625	7056	3969	2500	1764	1296	1024	784	625	289
8	62500	15625	7056	3969	2500	1764	1296	1024	784	625	289

% times a thread asked in vain for work (#waited for work / (#waited for work + # computational kernels))

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
2	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
4	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
6	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
8	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%

Speedup

Threads \ Tile size	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**150**
1	1.000	1.006	0.987	0.985	0.989	0.992	0.998	0.998	1.000	0.999	1.001
2	1.418	1.741	1.875	1.922	1.938	1.953	1.978	1.985	1.988	1.986	1.991
4	1.926	2.895	3.430	3.508	3.724	3.811	3.906	3.924	3.951	3.946	3.970
6	1.876	3.752	4.805	4.679	5.435	5.605	5.779	5.849	5.864	5.869	5.915
8	1.738	4.409	5.911	6.355	7.016	7.290	7.596	7.715	7.778	7.712	**7.838**

Table 28: PlusMultiply: 2500x2500 - collected and calculated statistics