Local Analysis of Scale Space

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Abstract

Scale space theory is a framework which permits one to analyze an image at different resolutions. The crucial feature is to let the resolution of an image vary continuously rather than discrete. This opens up the toolbox of mathematical analysis for image analysis. When extrema and saddle points of an image are tracked with increasing scale we know that they will have disappeared at sufficiently large scale. In this report we investigate what happens when extrema and saddles disappear. It turns out that extrema and saddles also appear for two dimensional images when scale is increased. Two dimensional images are more complicated than one dimensional signals, and some results generalize from two to higher dimensions. We consider two dimensional images as a compromise between generality and simplicity. One generic event is a saddle and an extremum merging and annihilating. Prior to annihilation they approach each other from opposite directions. The other generic event is the creation of a saddle-extremum pair. It will be shown that these are the only generic events. Studying scale space events for images in an image sequence or for a symmetric image, one discovers that more complicated interactions between extrema and saddles take place when scale is increased. The situations which arise when extrema and saddles disappear and appear can be classified by their codimension. The codimension is a measure of complexity. Generic events are the least complex and have codimension one. We also analyse events of codimension two. The mathematical tool which will be used for computing the local structure of toppoints is differential geometry of curves. A short version of this report was published in [8].

Introduction

A digital image is routinely represented as a rectangular lattice of integer numbers. This representation is inadequate for pattern recognition. A slight modification of the imaged object in one of two identical images (for instance by translation, rotation or scaling of the imaged object) changes the representation drastically. Scale space is intended to provide a better representation [6] [7] [10]. This alternative representation should be uncommitted in the sense that it must be possible in principle to reconstruct the lattice representation.

The idea is to select the generic events in scale space as the atoms of a representation. They can be described as follows. A *critical point* is a point in which the gradient is zero. The critical points are the extrema (maxima and minima) and the saddles of the image's grey level function, which is denoted L. A *critical curve* is a curve in scale space consisting of critical points. The Hessian H is the matrix whose elements are the partial derivatives of second order

$$H(x,y) = \left\{ \begin{array}{cc} L_{xx}(x,y) & L_{xy}(x,y) \\ L_{yx}(x,y) & L_{yy}(x,y) \end{array} \right\}$$

Partial derivatives are denoted by lower indices, for example L_{xx} denotes $\frac{\partial^2 L}{\partial x^2}$. The directions of maximum and minimum curvature are the eigenvectors of the Hessian. A critical point is a *toppoint* if at least one eigenvalue is zero, i.e. if the Hessian does not have full rank. It will be shown that the toppoints are precisely the points at which the generic events take place. The critical curves are solutions to the equations which expresses that the gradient is zero:

$$\begin{aligned} L_x(x,y,t) &= 0 \\ L_y(x,y,t) &= 0, \end{aligned} \tag{1}$$

where x and y denote the spatial coordinates, σ is scale and $t = 2\sigma^2$. In order to study this system of curves, it is useful to find an expression for the tangent at a given point. As is usual in differential geometry, we let s be a monotonically varying parameter along the curve. Thus x and y are functions of s and in this way L and its partial derivates with respect to x, y and t are implicit functions of s. It follows from equation (1), that for all integer n

$$\begin{aligned} \frac{d^n}{ds^n} L_x(x(s), y(s), t(s)) &= 0 \\ \frac{d^n}{ds^n} L_y(x(s), y(s), t(s)) &= 0. \end{aligned} \tag{2}$$

In what follows we shall use dots to denote differentiation with respect to s, such that for instance $\ddot{x}$ denotes $\frac{d^2 x}{ds^2}$. For $n = 1$ we get two equations using the chain rule.

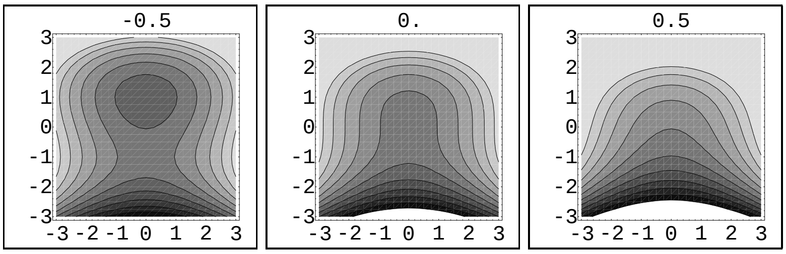

Figure 1: $L = 2x^2 + y^3 + 4t + 6yt$. The convex generic event. An extremum and a saddle merge when scale is increased. (left) $t = -5/10$. (center) $t = 0$. (right) $t = 5/10$.

$$\begin{aligned}
L_{xx}(x,y,t)\dot{x} + L_{xy}(x,y,t)\dot{y} + L_{xt}(x,y,t)\dot{t} &= 0 \\
L_{xy}(x,y,t)\dot{x} + L_{yy}(x,y,t)\dot{y} + L_{yt}(x,y,t)\dot{t} &= 0,
\end{aligned}$$

from which the directions of the tangent $(\dot{x}, \dot{y}, \dot{t})$ can be found at any point on the curve.

We shall analyse the curve at a particular point (x_0, y_0, t_0) by computing the tangent and its derivatives. In the following we will omit reference to (x_0, y_0, t_0) and by for instance L_{xx} understand $L_{xx}(x_0, y_0, t_0)$. We observe that if the Hessian has full rank, $\dot{t}$ can be used as a parameter along the tangent, meaning that the tangent is not horizontal, which in turn has the geometrical interpretation, that the critical point in the image does *not* disappear when the scale is infinitesimally increased. On the other hand, if the Hessian has rank 1 (still assuming the equations to be independent), the tangent is horizontal. The geometrical interpretation is that two critical points merge and disappear when the scale is infinitesimally increased.

The event just described is unavoidable for gaussian smoothing. Critical points must disappear with increasing scale and this event is the generic way in which it happens. Figure 1 shows an example. It is visualized by the contour curves of the image at scales below, at, and above the scale of the toppoint.

The term generic means that if the image is changed slightly, the event may change position in scale space, but it will still be present. Perhaps surprisingly, also another event is generic: the *appearing* of two critical points. Below we shall derive the conditions for this to happen. These two events are the only generic ones, and are among the primitives proposed for image representation.

The next three sections analyze in detail what happens at codimension one and two. Then follows description of a setup, which allows one to compute the local scale space behaviour at any point of an image. Suggestions for further research terminate the paper.

Codimension

Even if the two above events are sufficient to describe a typical image, in three situations it is required to study degenerate cases in which the image can not be said to be typical. These include *local symmetry*, *quantization* of scalar and spatial resolution, and temporal *image sequences*. In all three cases the additional image constraints have the effect that more than two points may be involved in a generic scale space event. Thus, it will be necessary to include degenerate events as description primitives. Below tools for describing degenerate events are put forward. In order to understand the algebra involved, let us rewrite the equations (2) in which the partial derivatives have been evaluated at (x_0, y_0, t_0) for $n = 1$ and $n = 2$.

$$\begin{aligned} L_{xx}\dot{x} + L_{xy}\dot{y} + L_{xt}\dot{t} &= 0 \\ L_{xy}\dot{x} + L_{yy}\dot{y} + L_{yt}\dot{t} &= 0 \end{aligned} \tag{3}$$

$$\begin{aligned} L_{xxx}\dot{x}^2 + L_{xyy}\dot{y}^2 + L_{xtt}\dot{t}^2 + 2L_{xxy}\dot{x}\dot{y} + 2L_{xxt}\dot{x}\dot{t} + 2L_{xyt}\dot{y}\dot{t} + L_{xx}\ddot{x} + L_{xy}\ddot{y} + L_{xt}\ddot{t} &= 0 \\ L_{yxx}\dot{x}^2 + L_{yyy}\dot{y}^2 + L_{ytt}\dot{t}^2 + 2L_{xyy}\dot{x}\dot{y} + 2L_{xyt}\dot{x}\dot{t} + 2L_{yyt}\dot{y}\dot{t} + L_{xy}\ddot{x} + L_{yy}\ddot{y} + L_{yt}\ddot{t} &= 0. \end{aligned} \tag{4}$$

If we impose one algebraic constraint, then we say that the event described has *codimension* one. We choose the constraint that the rank of H is one. An immediate consequence is that the tangent is horizontal (use equations (3)). From the same equations the tangent direction can be found.

If we impose an additional constraint the codimension becomes two. Codimension two events are determined by the additional constraint that the Hessian has rank zero. In this case equations (3) still imply that a tangent must be horizontal but they are no longer sufficient to determine the tangent direction. We must invoke equations (4). Since the Hessian has rank zero, $\ddot{t}$ can be eliminated to get a quadratic equation in the unknowns $\dot{x}$ and $\dot{y}$. It turns out that for Gaussian scale space there are two horizontal perpendicular tangents in a codimension two event. The geometric interpretation is that two critical curves intersect, and the solutions provide the direction of the tangents at the point of intersection. Figure 2 shows two extrema and two saddles which disappear simultaneously.

Please note that the computations are undertaken projectively. Apparently we have one equation too few for the tangent and for its higher derivatives. If we insist on computing the exact size of the tangent and its derivatives, we can use a set of equations which expresses that the hypothetical point traversing the curve has constant velocity. This means that the parameter s measures

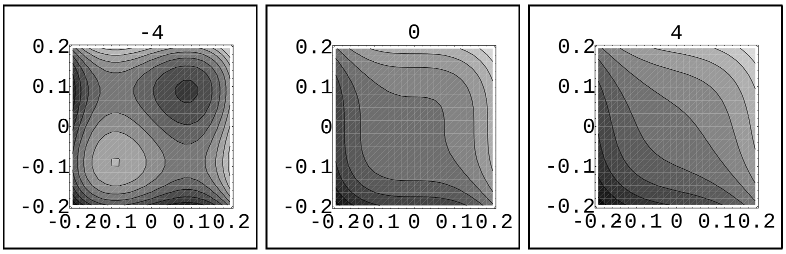

Figure 2: $L = x^3 + y^3 + 6(x + y)t$. Two extrema and two saddles disappear simultaneously when scale is increased. The two saddles merge from opposite directions simultaneously with two extrema (a maximum and a minimum) merging from the perpendicular direction. Extrema enter from north-east and south-west and saddles from north-west and south-east. (left) $t = -4/1000$. (center) $t = 0$. (right) $t = 4/1000$.

arclength. The first few equations are

$$\begin{aligned} \dot{x}^2 + \dot{y}^2 + \dot{t}^2 &= 1 \\ 2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} + 2\dot{t}\ddot{t} &= 0 \\ 2\,\ddot{x}^2 + 2\,\ddot{y}^2 + 2\,\ddot{t}^2 + 2\,\dot{x}\,\dddot{x} + 2\,\dot{y}\,\dddot{y} + 2\,\dot{t}\,\dddot{t} &= 0. \end{aligned}$$

Analysis of codimension one

Let the rank of H be one. (L_{xx}, L_{xy}) and (L_{xy}, L_{yy}) cannot vanish simultaneously. Let $(L_{xx}, L_{xy}) \neq (0, 0)$. $\dot{x}$ and $\dot{y}$ can be eliminated from equation (3) to get

$$\begin{vmatrix} L_{xx} & L_{xt} \\ L_{xy} & L_{yt} \end{vmatrix} \dot{t} = 0,$$

and we may infer that $\dot{t} = 0$, which implies horizontal tangent. The tangent direction can then be computed from equation (3).

In order to know if the curve is convex (two points disappear with increasing scale) or concave (two points appear with increasing scale) we must compute $\ddot{t}$. In the convex case it is negative and it is positive in the concave case. We wish further to know which types (saddle or extrema) are involved. To this end we need to know the sign of the Hessian determinant on the curve. To find $\ddot{t}$ we can eliminate $\ddot{x}$ and $\ddot{y}$ from equation (4) to get

$$\begin{vmatrix} L_{xxx} & L_{xx} \\ L_{xxy} & L_{xy} \end{vmatrix} \dot{x}^2 + 2 \begin{vmatrix} L_{xxy} & L_{xx} \\ L_{xyy} & L_{xy} \end{vmatrix} \dot{x}\dot{y} + \begin{vmatrix} L_{xyy} & L_{xx} \\ L_{yyy} & L_{xy} \end{vmatrix} \dot{y}^2 + \begin{vmatrix} L_{xt} & L_{xx} \\ L_{yt} & L_{xy} \end{vmatrix} \ddot{t} \;=\; 0.$$

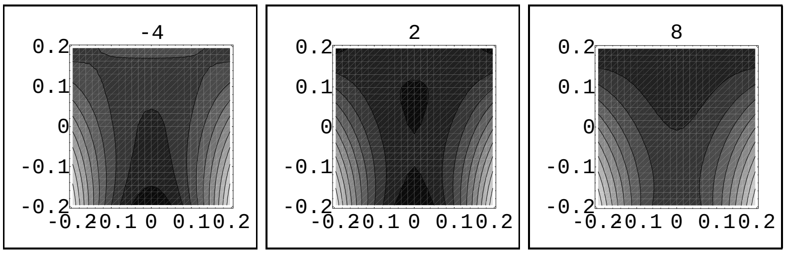

Figure 3: $L = 2x^2 - 12x^2y + 2y^3 - 12yt + 4t$. The concave generic event. An extremum and a saddle split. (left) $t = -4/1000$. (center) $t = 2/1000$. (right) $t = 8/1000$.

Depending on the values of the partial differential quotients at the toppoint, $\ddot{t}$ can be positive or negative. This means that the curve may be convex or concave. An example of a generic convex event is shown in figure 1, and figure 3 shows an example of a concave generic event.

Next we want to know the type of points involved. An approximation to first order in s of the Hessian is computed on each of the two tangents. This is done by approximating the partial derivatives on the curve to first order in s.

$$\begin{vmatrix} L_{xx}(x,y,t) & L_{xy}(x,y,t) \\ L_{yx}(x,y,t) & L_{yy}(x,y,t) \end{vmatrix} = \begin{vmatrix} L_{xx} + s(L_{xxx}\dot{x} + L_{xxy}\dot{y}) & L_{xy} + s(L_{xxy}\dot{x} + L_{xyy}\dot{y}) \\ L_{yx} + s(L_{xxy}\dot{x} + L_{xyy}\dot{y}) & L_{yy} + s(L_{xyy}\dot{x} + L_{yyy}\dot{y}) \end{vmatrix} = O(s).$$

This means that the sign changes as s passes through the toppoint. The geometric interpretation is that a saddle and an extremum are involved. This is true for both a convex and a concave curve.

Analysis of codimension two

Let us next assume that the rank of the Hessian is 0, which means that $L_{xx} = L_{xy} = L_{yy} = 0$. It follows from equation (3) that $\dot{t} = 0$, which implies that the tangents are horizontal. I use the plural here since there are actually two tangents. This can be seen from equations (4) from which $\ddot{t}$ has been be eliminated.

$$\begin{vmatrix} L_{xxx} & L_{xt} \\ L_{xxy} & L_{yt} \end{vmatrix} \dot{x}^2 + 2 \begin{vmatrix} L_{xxy} & L_{xt} \\ L_{xyy} & L_{yt} \end{vmatrix} \dot{x}\dot{y} + \begin{vmatrix} L_{xyy} & L_{xt} \\ L_{yyy} & L_{yt} \end{vmatrix} \dot{y}^2 \quad = \quad 0. \qquad (5)$$

We notice that in the case of gaussian smoothing (use the relations $L_{xt} =$

$L_{xxx} + L_{xyy}$ and $L_{yt} = L_{xxy} + L_{yyy}$ which are induced by the heat equation)

$$\begin{vmatrix} L_{xxx} & L_{xt} \\ L_{xxy} & L_{yt} \end{vmatrix} + \begin{vmatrix} L_{xyy} & L_{xt} \\ L_{yyy} & L_{yt} \end{vmatrix} = 0.$$

Since the coefficients of $\dot{x}^2$ and $\dot{y}^2$ have the same magnitude but different sign, the discriminant of the quadratic equation is positive and two real solutions exist for the tangent directions. A further consequence is that the two directions are perpendicular to each other. This follows by observing that the product of the roots is -1.

$\ddot{t}$ can be found from equation (4). It is further seen that $|H| = O(s^2)$ and we infer that along each of the two intersecting curves two critical points of the same type approach each other. In order to gain more insight in the structure we now orient the coordinate system such that the tangents point along the x-axis and the y-axis. This has an equivalent formulation in terms of a constraint between the partial derivatives. It can be found from equation (5) which was solved for the two tangent directions. The coefficient of the quadratic term must be zero. This means that $L_{xxx}L_{yyy} - L_{xxy}L_{xyy} = 0$. The Hessian computed on the tangent which points along the y-axis becomes

$$H_1 = H(0, s) = (L_{xxy}L_{yyy} - L_{xyy}^2)s^2.$$

H on the tangent directed along the x-axis is

$$H_2 = H(s, 0) = (L_{xxx}L_{xyy} - L_{xxy}^2)s^2.$$

Notice that

$$L_{xxy}H_1 + L_{yyy}H_2 = 0.$$

We shall also need $\ddot{t}$ on the two tangents. Let $\ddot{t}_1 = \ddot{t}(0, s)$ be $\ddot{t}$ on the tangent directed along the y-axis, and let $\ddot{t}_2 = \ddot{t}(s, 0)$ be $\ddot{t}$ on the tangent directed along the x-axis.
From equation 4 we get

$$\ddot{t}_1 = -L_{yyy}/\left(L_{xxy} + L_{yyy}\right)$$

and

$$\ddot{t}_2 = -L_{xxy}/\left(L_{xxy} + L_{yyy}\right).$$

Notice that

$$\ddot{t}_2 H_1 + \ddot{t}_1 H_2 \quad = \quad 0. \tag{6}$$

Case 1a: $L_{xxy}L_{yyy} > 0$. L_{xxy} and L_{yyy} have same sign. Both $\ddot{t}_1$ and $\ddot{t}_2$ are negative, which means that both curves are convex. Equation (6) implies that

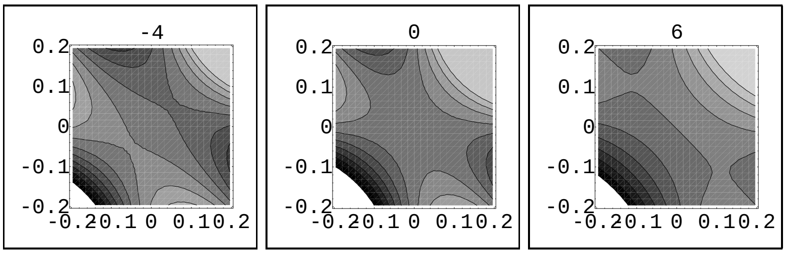

Figure 4: $L = x^2y + y^2x + 2(x+y)t$. Two saddles merge from opposite directions and separate in the perpendicular direction. They enter from north-east and south-west and separate in directions north west and south east. (left) $t = -4/1000$. (center) $t = 0$. (right) $t = 6/1000$.

H_1 and H_2 have opposite signs, which means that one curve consists of extrema and the other curve consists of saddles. Figure 2 shows an example.

Case 1b: $L_{xxy}L_{yyy} < 0$. L_{xxy} and L_{yyy} have opposite signs. It is immediately seen that H_1 is negative. $\ddot{t}_1\ddot{t}_2 < 0$ which means that one curve is convex and the other concave. Equation 6 implies that H_2 is also negative. Consequently both curves consist of saddles. Figure 4 shows an example.

Two critical curves may intersect with non-horizontal tangents [3]. An example is shown in figure 5.

Computing the local graph

Below follows directions for computing the local graph at a point in an image. It can be visualized for instance by the contour curves at different values of t as illustrated by the figures, which have been computed from the local jet.

Heat Polynomials

Once we know the n-jet, we further need to know the dependence on scale. This dependence is supplied by the *heat polynomials* [9]. The heat polynomial $v_n(x,t)$ is a solution to the heat equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{df}{dt}$$

with the initial condition x^n. The first few heat polynomials are

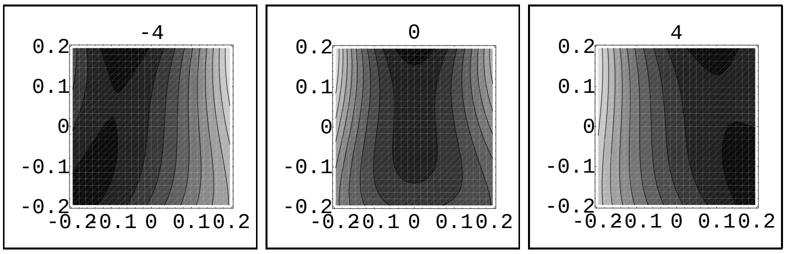

Figure 5: $L = x^2 - x^3 + 3x^2y - y^3 - x^4y + 2t - 6tx - 12t^2y - 12tx^2y$. Two non-horizontal intersecting tangents. For negative t a saddle approaches from the second quadrant at the same time as a maximum from the third quadrant. For $t = 0$ they form a shoulder. For positive t it is split into a saddle running up into the first quadrant and a maximum into the fourth quadrant. The two critical curves intersect. (left) $t = -4/10$. (center) $t = 0$. (right) $t = 4/10$.

$$\begin{aligned}
v_0(x,t) &= 1 \\
v_1(x,t) &= x \\
v_2(x,t) &= x^2 + 2t \\
v_3(x,t) &= x^3 + 6xt \\
v_4(x,t) &= x^4 + 12x^2t + 12t^2
\end{aligned}$$

Their recursion formula is

$$v_n(x,t) = xv_{n-1}(x,t) + 2t(n-1)v_{n-2}(x,t). \tag{7}$$

The solution to the heat equation for initial condition $x^n y^m$ is $v_n(x,t)v_m(y,t)$ and is called the *extension* of $x^n y^m$ [1]. As a consequence the Taylor expansion in the variables x, y and t can be found from the Taylor expansion of the image in the variables x and y by substituting $v_n(x,t)v_m(y,t)$ for each monomial $x^n y^m$. As an example, the polynomial

$$2x^2 - 12x^2y + 2y^3$$

is extended to

$$2(x^2 + 2t) - 12(x^2 + 2t)y + 2(y^3 + 6yt) = 2x^2 - 12x^2y + 2y^3 - 12yt + 4t$$

Another useful relation is $\frac{\partial v_n(x,t)}{\partial x} = nv_{n-1}(x,t)$. Its consequence is that we may take the gradient before we extend the local jet.

The appendix lists a Mathematica [11] function which computes the heat polynomials.

Global analysis

What happens with the critical points created by a concave generic event? Using a polynomial to represent the jet, the critical points may escape to infinity. A set-up can be used which is similar to the technique which has been introduced. A Fourier series is used rather that a polynomial. That representation has the property that points do not disappear to infinity, but stay in a bounded area.

The first step to study the global behavior of a singularity is to compute its partial derivatives, its jet, to a certain order. Next a Fourier series must be found which has the same jet at the toppoint up to the chosen order. Then the Fourier series must be extended to include explicit dependence on t to be able to perform analytical computations. This is the easy part. The solution to the heat equation for initial condition

$$e^{inx} = \cos(nx) + i\sin(nx)$$

is

$$e^{-n^2 t} e^{inx} = e^{-n^2 t}\cos(nx) + ie^{-n^2 t}\sin(nx).$$

The appendix contains a listing of a Mathematica function to construct a Fourier series to represent the jet. One may then experimentally follow the development for increasing scale by graphical means.

Harmonic Polynomials Harmonic polynomials can be used as a basis rather than monomials [1]. Let H_m be the homogeneous harmonic polynomials of degree m. For two spatial variables H_m consists of two polynomials. Let $r^2 = (x^2 + y^2)$. A Basis for homogeneous polynomials of degree m is

$$H_m \oplus r^2 H_{m-2} \oplus r^4 H_{m-4} \oplus \ldots$$

H_m are computed recursively as follows $H_0 = \{h_{0,1}, h_{0,2}\} = \{1, 0\}$. Let $H_m = \{h_{m,1}, h_{m,2}\}$.

$$(h_{m+1,1}, h_{m+1,2}) = (xh_{m,1} - yh_{m,2}, xh_{m,2} + yh_{m,1}) \tag{8}$$

The first few H_m are

$$\begin{aligned}
H_0 &= \{1\} \\
H_1 &= \{x, y\} \\
H_2 &= \{x^2 - y^2, xy, x^2 + y^2\} \\
H_3 &= \{x^3 - xy^2, y^3 - x^2y, x(x^2 + y^2), x(x^2 + y^2)\}
\end{aligned}$$

Since the harmonic polynomials per definition are in the kernel of the Laplacian, they are not augmented by terms including powers of t when extended to heat polynomials. Their local structure is independent of t. This separation into t-dependent and t-dependent basis functions may yield equations that are easier to analyze.

Suggestions for Further Research

Local Symmetry Describe the degenerate toppoints by a symmetry group. A generic top point has a 180° symmetry when simultaneously interchanging extremum and saddle. Codimension two events have larger symmetry groups. Can one assign a unique group to a toppoint? Which groups are they? Does a family of groups characterize gaussian scale space?

Image Representation Is it possible and useful to represent an image by its toppoints? In one dimension a representation by toppoints has been proposed [4]. Knowledge of the position of toppoints in scale space is sufficient. It is not immediately transferable to 2 dimensions because concave events are generic. Convex toppoints are unavoidable. Are concave toppoints also unavoidable? How frequent are they?

Toppoint Classification Describe the structure of toppoints with high codimension for gaussian scale space. Presumably it can be done by considering the general form of solutions to the equations for the tangent and its derivates. Bezout's theorem [2] which states that a homogeneous system of equations has a number of solutions equal to the product of the equations' degrees must be essential. Also the formula for n times implicit differentiation will be needed. It is called Faa di Bruno's formula and can be found in [5].

Acknowledgement

Luc Florack gave me comments on the final draft. Jørgen Sand gave me advice on Fourier Series.

Appendices

A Mathematica function for heat polynomials

```
(* *****************
HeatPolynomial[pol,variables] has a homogeneous polynomium as input
and a list containing the variables. The result is the corresponding
heat polynomial in which t is the time variable.
Litt:   James Damon: Local Morse Theory for Solutions to the Heat
  Equation and Gaussian Blurring, in Journal of
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                  Differential Equations 1995,Vol 115 pages 368-401.
D. V. Widder: The Heat Equation
*)
HeatPolynomial[pol_,variables_]:=
Module[{auxpol1},
auxpol1=Expand[pol];
  (* Et udtryk i +normalform(efter Expand) er et tr med + i roden
hvis der er flere termer. Der er et * eller ^ i toppen hvis der kun er
en term som bestr af to faktorer. Ellers er det vist et atom hvis det
er en konstant.*)
If[(Head[auxpol1]=!=Plus)
,
(* Only one term. Pass the Buck *)
  HeatPolynomialSingleTerm[pol,variables]
,
  (* More terms. Do recursion *)
  HeatPolynomialSingleTerm[First[auxpol1]        ,variables]+
  HeatPolynomial          [auxpol1-First[auxpol1],variables]
]
(* end If *)
]
(* end module HeatPolynomial *)

(* *****************
HeatPolynomialSingleTermSingleVariable[pol,variables] has a
homogeneous polynomium in one variable as input and a list with the
variable as only element. t is the time variable. The result is the
derived heat polynomial.
 *)
HeatPolynomialSingleTermSingleVariable[pol_,variables_]:=
Module[{auxpol1,expx,expt},
  auxpol1=Expand[pol];
    expx=Exponent[auxpol1,First[variables]];
    expt=Exponent[auxpol1,t];
    If[((auxpol1===0)||(expx===0)||(expx===1))
    ,
     auxpol1
    ,
     (* Only one term degree(x) at least 2. Do the real work*)
       auxpol1+
       HeatPolynomialSingleTermSingleVariable[
              Expand[auxpol1/First[variables]/First[variables]*
                     expx*(expx-1)*t/(expt+1)],
    ]
```

```
  (* end If *)
  ]
(* end module HeatPolynomialSingleTermSingleVariable *)

(* *****************
HeatPolynomialSingleTerm[pol,variables] has a single additive term as
input and a list containing the variables. t is the time variable.
*)
HeatPolynomialSingleTerm[pol_,variables_]:=
Module[{auxpol1},
auxpol1=Expand[pol];
If[(SameQ[variables,{}]||(auxpol1===0))
 ,(* No more variables or zero. Copy to output *)
 auxpol1
 ,(* The heatpolynomials are separable: heat[x^n y^m]=heat[x^n]heat[y^m].
     Separate out the first variable *)
  HeatPolynomialSingleTermSingleVariable[
          First[variables]^Exponent[auxpol1,First[variables]],
          {First[variables]}]*
  HeatPolynomialSingleTerm[
          auxpol1/First[variables]^Exponent[auxpol1,First[variables]],
                  Drop[variables,1]]
]
(* end If *)
]
(* end module HeatPolynomialSingleTerm *)
```

A Mathematica function for the representation of a jet as Fourier Series

```
(* *****************
Local2Global[pol,{x,y},n] computes a Fourier Series with the same jet up to
order n(n even) at (x,y)=(0,0)  as pol
*)
Local2Global[pol_,{x_,y_},n_]:=Module[{first},
first=Local2GlobalOneVariable[pol,x,n];
ComplexExpand[Expand[Local2GlobalOneVariable[first,y,n]]]
](*end Global2Local *)

(* *****************
  Local2GlobalBase[x_,n_] computes the Fourier Series with the same
  jet up to order n(n even) at x=0 for the polynomials
  (1,x^2,x^3,x^4, ... , x^n)
*)
```

```
Local2GlobalBase[x_,n_]:=Module[{res,halfn,matrixn,invmatrixn,trigxn,fac},
halfn=Floor[n/2];
matrixn=Table[D[Exp[I j x],{x,i}],{i,0,n},{j,-halfn,halfn}]/.x->0;
matrixn=Transpose[
     Times[Transpose[matrixn],Table[Exp[(t i^2)],{i,-halfn,+halfn}]]];
(* Compute the derivatives at x=0 for each of the complex exponentials*)
invmatrixn=Inverse[matrixn];
(* Inverting the matrix yields coefficients for  (1,x^2,x^3,...,x^n)*)
trigxn=Table[Exp[I j x],{j,-halfn,halfn}];
fac=Table[(j!),{j,0,n}];
res=Dot[Transpose[invmatrixn],trigxn];
Times[fac,res]
](* end Local2GlobalBase*)

(* *****************
  Local2GlobalOneVariable[pol,x,n] computes a Fourier Series with
  the same jet up to order n(n even) at x=0  as pol
*)
Local2GlobalOneVariable[pol_,x_,n_]:=Module[{list,poly,newpoly},
list=Local2GlobalBase[x,n];
(*list of expressions to substitute for 1,x,x^2, ...*)
poly=Expand[pol];
newpoly=Expand[Dot[list,Table[Coefficient[poly,x,i],{i,0,n}]]]
](* end Local2GlobalOneVariable*)
```

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